

Studies of Solar Granulation:

I. The Statistical Interpretation of Granule Structure from One-Dimensional Microphotometer Tracings

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Introduction

Photometric data derived from photographs showing solar granulation have recently been published by Frenkiel and Schwarzschild (1955) and by Uberoi (1955a). These authors have presented the autocorrelation function of brightness of the photographs and the frequency spectrum of the brightness distribution. Their work reveals that the large-scale fluctuations are apparently highly important and that in the small-scale range there is no particular size much more probable than the others. This last point contradicts a result obtained by Stuart and Rush (1954) from data of Richardson and Schwarzschild (1950), who collected data on the change of brightness and on the Doppler shift, using a high quality spectrum of solar granulation. Stuart and Rush computed from these published data the autocorrelation function of velocity and the cross-correlation function of velocity and brightness, using moving averages to attenuate large-scale fluctuations. Because these two functions presented a definite peak corresponding to a wavelength of 2300 km, these authors concluded that "the spectrum of velocities for the small granules consists of a *narrow*

distribution of cell sizes centered about an average size of 2300 km." To discover why Stuart and Rush obtained a result different from that of the other authors, I have analyzed new tracings of solar granulation with and without the use of the moving average. The results are summarized below. I have also compared visual and photometric measurements of granulation photographs and have shown why they give different results.

Observational material

For this analysis I have used an excellent granulation photograph taken by Dr. W. A. Miller at R. C. A. Laboratories, Rocky Point, Long Island, New York, on Apr. 5, 1953. The aperture of the objective was 11.3 cm, the plate Kodak 548 GH. No filter was used. The diameter of the solar image was 11.2 cm. With the microphotometer of the High Altitude Observatory, I made a tracing of the central region of the sun. The record covered 220,000 km. The slit of the photometer was a rectangle of 18 by 24 microns, equivalent to a solar area of 225 by 300 km. On the recording paper, 1 cm corresponded to 730 km on the sun. The paper had a

¹ High Altitude Observatory of the University of Colorado, at time of this study. Now at the Observatoire de Paris.

linear scale 0-100 and the deflections T_i were read to the nearest integer at 715 equidistant intervals, each elementary interval being equivalent to the distance $D=308$ km. Since the plate was not calibrated, the absolute values of the brightness were not known. However, measurements with similar calibrated plates indicate that there is a linear relationship between changes in brightness that are relatively small and the deflections of the recorder. Since my measurements are in a small range, I have assumed that the deflections are proportional to brightness.

Autocorrelation functions without and with moving average

To compute the autocorrelation function up to a spacing of about 20,000 km (equivalent to 65 elementary intervals D of 308 km), the 715 deflections T_i were handled as follows: I computed the following 66 population means:

$$\bar{T}_{i+m} = \frac{1}{650} \sum_{i=1}^{650} T_{i+m}, \quad (1)$$

where m is an integer: $m=0, 1, 2, \dots, 64, 65$. Then, I computed the deviations from the mean:

$$X_{i+m} = T_{i+m} - \bar{T}_{i+m}. \quad (2)$$

Finally, I computed the autocorrelation coefficient $B(mD)$, without moving average, as a function of the spacing mD on the sun, by the formula:

$$B(mD) = \frac{\sum_{i=1}^{650} X_i X_{i+m}}{\left(\sum_{i=1}^{650} X_i^2 \sum_{i=1}^{650} X_{i+m}^2 \right)^{1/2}}. \quad (3)$$

Figure 1 shows the autocorrelation function $B(mD)$. After a steep decrease to $m=7$, the function presents large-scale oscillations. Other tracings of the same plate indicate that the peaks at $m=29$ and $m=40$ correspond to statistical fluctuations only. This result will be discussed in a later paper.

I also computed for the 715 i 's the moving average A_i of 9 consecutive deflections and the deviations Y_i from this moving average, according to the formulae:

$$A_i = \frac{1}{9} \sum_{n=-4}^{+4} T_{i+n}, \quad (4)$$

$$Y_i = T_i - A_i. \quad (5)$$

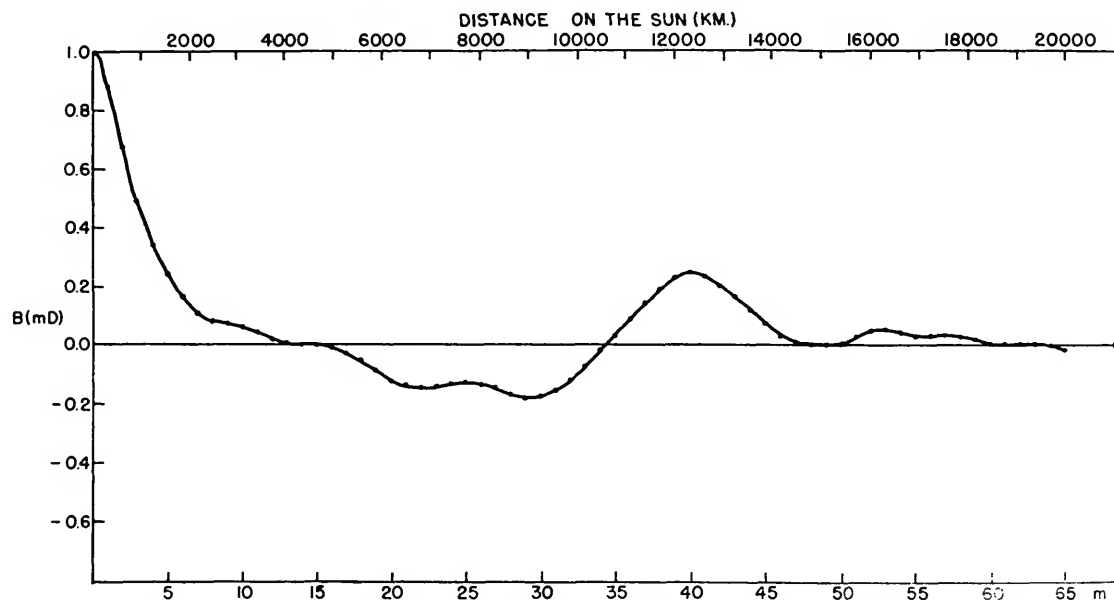


FIGURE 1.—Autocorrelation function of brightness from direct readings without moving average.

As the distance between two successive readings is equivalent to $D=308$ km, A_i is an average over a small interval of about 2,800 km. Figure 2 shows the autocorrelation function

$B_d(mD)$ corresponding to the deviations Y_i and computed by a formula identical to (3). The curve presents more or less regular oscillations around zero. It has 8 maxima and the

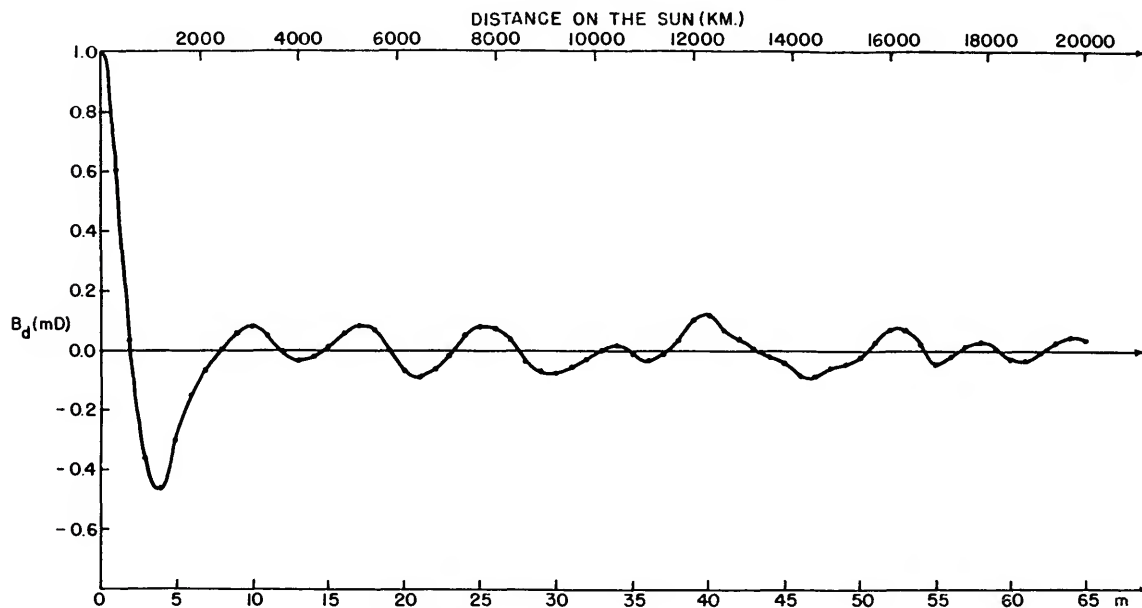


FIGURE 2.—Autocorrelation function of brightness by using deviations from the moving average of 9 points.

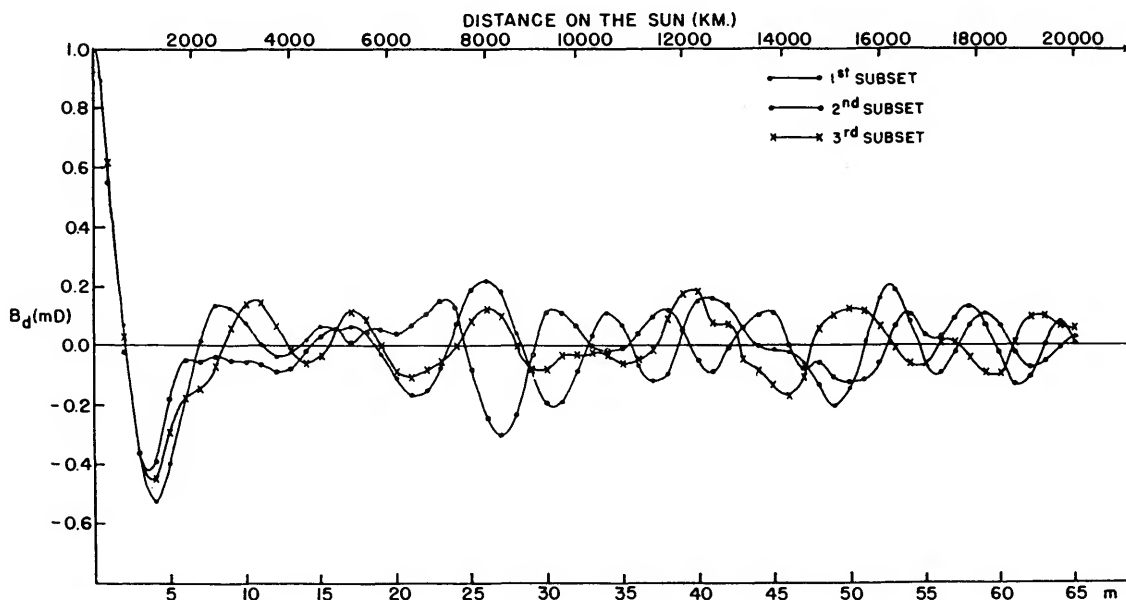


FIGURE 3.—Autocorrelation function of brightness by using deviations from the moving average of 9 points. Data divided into three equal subsets.

mean distance between them is $8D$, about 2,500 km. The curve has also a very strong minimum around $m=4$, at about 1,200 km. With the exception of this minimum, the values of $B_d(mD)$ for successive extremes have an irregular variation and in any case do not show a clear decrease when m increases. Such a decrease, on the other hand, should occur if the oscillations were due to a correlation resulting from a well-defined cell size on the tracing.

To appreciate more precisely the significance of the successive extremes of figure 2, I divided the set of data into three equal subsets and computed for each one the autocorrelation function of brightness using the moving average. The results, plotted on figure 3, show three facts:

(a) The three curves present the same pseudo-periodicity and the same general character as the curve of figure 2, although the extremes of $B_d(mD)$ have a greater absolute value, corresponding to the fact that the statistical samples are smaller.

(b) For $m > 6$, thus for a spacing larger than 1,800 km, the extremes of any one curve are randomly dis-

tributed with respect to the extremes of the two other curves.

(c) Up to $m=3$, the three curves coincide.

Points (a) and (b) show that the oscillations are not the result of well-defined cells but are only statistical fluctuations having a pseudo-period introduced by the moving average. For a very large sample, the absolute value of the extremes of $B_d(mD)$ would probably be smaller, with the exception of the first minimum which would be only slightly affected. Point (c) raises a question: Is the beginning of the curve up to the minimum at $m \approx 4$ significant?

To clarify this point, I used the moving average of 17 points for the first subset. The corresponding autocorrelation function, plotted in figure 4, shows that the first minimum appears now for $m \approx 8$ and that the pseudo-periodicity has a scale twice as great as in figures 2 and 3.

Thus, clearly, the shape of figure 2 is mainly determined by the characteristics of the moving average used and by the fact that the sample

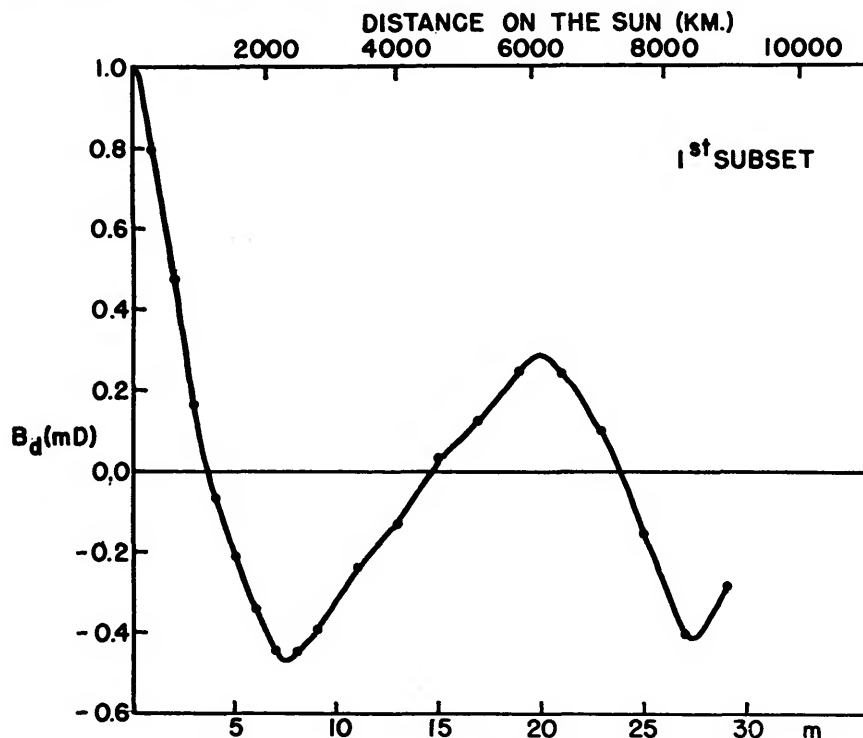


FIGURE 4.—Autocorrelation function of brightness by using the deviations from the moving average of 17 points.

was too small for the positions and values of the extremes of the curve to be statistically significant.

Mathematical formulation

These results can be confirmed theoretically by using methods and general formulae derived by Doob (1951). The application of these formulae is given below in the particular case of the moving average of 9 points used in figures 2 and 3.

Autocorrelation function.— $E(mD)$, $Q(mD)$ and $S(mD)$ are called the covariances of X_t , Y_t and $A_t - \bar{T}_t$. The sample is supposed to be very large and to be stationary, so that $\bar{T}_{t+m} = \bar{T}_t$. By definition:

$$E(mD) = \frac{1}{N} \sum_{i=1}^N X_i X_{i+m}, \quad (6)$$

$$Q(mD) = \frac{1}{N} \sum_{i=1}^N Y_i Y_{i+m}, \quad (7)$$

$$S(mD) = \frac{1}{N} \sum_{i=1}^N (A_i - \bar{T}_i)(A_{i+m} - \bar{T}_i), \quad (8)$$

where N is the number of deflections involved in the means, here supposed to be very large.

Formulae (2), (4), and (5) give the equations:

$$Y_t = \frac{1}{9} \left(-\sum_{n=1}^4 X_{t-n} + 8X_t - \sum_{n=1}^4 X_{t+n} \right), \quad (9)$$

$$Y_{t+m} = \frac{1}{9} \left(-\sum_{n=1}^4 X_{t+m-n} + 8X_{t+m} - \sum_{n=1}^4 X_{t+m+n} \right). \quad (10)$$

Since N is a very large number, it is possible to express $Q(mD)$ as a linear combination of the covariance E . Let $m' = mD$; it then follows from equations (6) (7) (9) and (10) that:

$$\begin{aligned} Q(m') = & \frac{1}{81} [E(m'-8) + 2E(m'-7) + 3E(m'-6) \\ & + 4E(m'-5) - 13E(m'-4) - 12E(m'-3) \\ & - 11E(m'-2) - 10E(m'-1) + 72E(m') \\ & - 10E(m'+1) - 11E(m'+2) - 12E(m'+3) \\ & - 13E(m'+4) + 4E(m'+5) + 3E(m'+6) \\ & + 2E(m'+7) + E(m'+8)]. \end{aligned} \quad (11a)$$

This formula is valid even for $m' < 8$, if we assume that

$$E(-j) = E(j). \quad (12)$$

Thus, $Q(mD)$ can be derived from the covariances E if one uses a weighted moving average defined by:

$$\begin{aligned} Q = & \frac{1}{81} (1, 2, 3, 4, -13, -12, -11, \\ & -10, 72, . . .)E. \end{aligned} \quad (11b)$$

Similarly $S(m)$ can be derived from the covariances E if one uses a moving average defined by:

$$S = \frac{1}{81} (1, 2, 3, 4, 5, 6, 7, 8, 9, . . .)E. \quad (13)$$

Formulae (11) and (13) define linear arithmetical filters represented in figure 5, for the case where $m' > 8$. Formula (11) explains the properties of the curve of figure 2. It is instructive to apply it to a correlation function having no oscillatory character. For this I used:

$$B(m') = e^{-.068m'^2}. \quad (14)$$

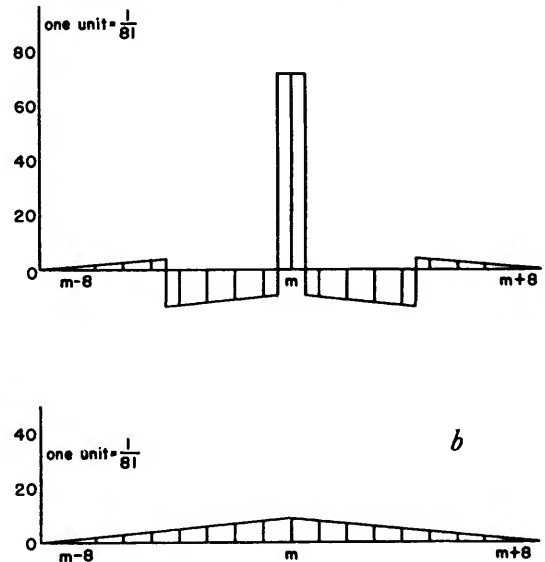


FIGURE 5.—Linear filters for the correlation function: *a*, for the deviations of the moving average of 9 points; *b*, for the moving average of 9 points itself.

Figure 6 shows $B(m')$ and its transform $B_d(m')$ after application of formula (11). One sees the oscillatory character of $B_d(m')$ and particularly the strong minimum around $m = 5$.

An analysis similar to that made here for the autocorrelation function of brightness could be made also for the cross-correlation function of velocity and brightness. It would likely show that the use of moving averages is similarly misleading.

Frequency spectrum.—Another way to evaluate the effects due to the use of the moving average is to compare the frequency spectra derived from the correlation functions of figures 1 and 2 by the use of a Fourier transform. The frequency spectrum $F(k)$ corresponding to the X_i 's of page 26 can be written:

$$F(k) = \frac{D}{\pi} \overline{X_i^2} \left[1 + 2 \sum_{m=1}^{64} B(mD) \cos \frac{hm\pi}{65} + B(65D) \cos(h\pi) \right] \quad (15)$$

where

$$k = \frac{2\pi}{\lambda} = \frac{h\pi}{65D},$$

and h is an integral number varying from 0 to 65, λ is a wavelength on the sun, k is the space

equivalent of a time pulsation (2π times a wavenumber). The spectrum $F_d(k)$, corresponding to the deviations from the moving average, can then be obtained from formula (15) by replacing $B(mD)$ by $B_d(mD)$. One can then show that for a very large sample the relation between $F(k)$ and $F_d(k)$ is given by:

$$F_d(k) = G_d^2(k) F(k), \quad (17)$$

where

$$G_d(k) = \frac{2}{9} (4 - \cos kD - \cos 2kD - \cos 3kD - \cos 4kD). \quad (18)$$

$G_d(k)$ is the gain of the filter defined by the moving average procedure. Figure 7 shows the two spectra for the same data used in page 26. The solid curve of figure 8 represents $G_d^2(k)$ and the crosses indicate the values obtained from figure 7. The departures of the crosses from the curve are partly due to the fact that the actual sample used was not large enough and partly due to numerical approximations in the computation.

The spectrum $F(k)$ is of the low frequency type, i. e., the average energy per unit frequency decreases with frequency. On the other hand, the spectrum $F_d(k)$ is more or less symmetrical

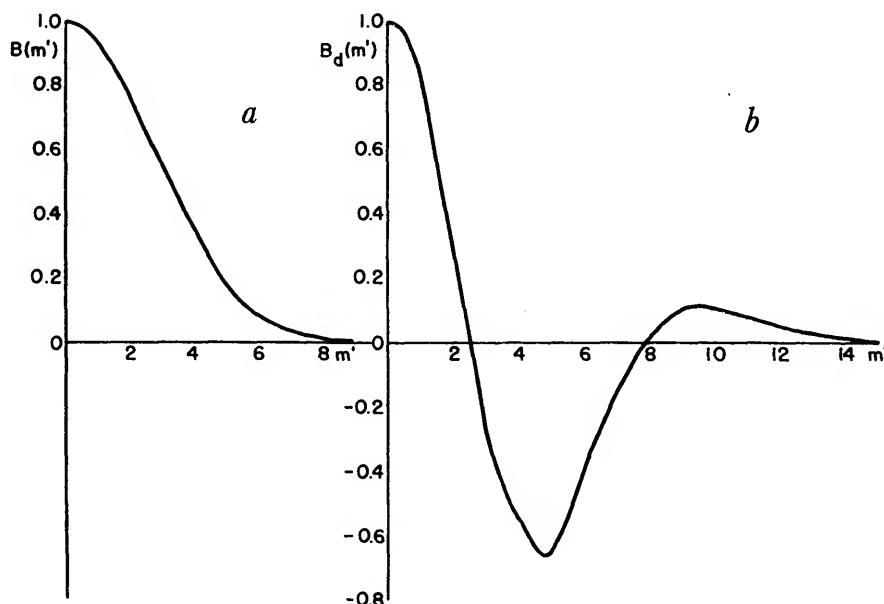


FIGURE 6.—*a*, Autocorrelation function having a non-oscillatory character; *b*, its transform by use of the filter $Q(m')$

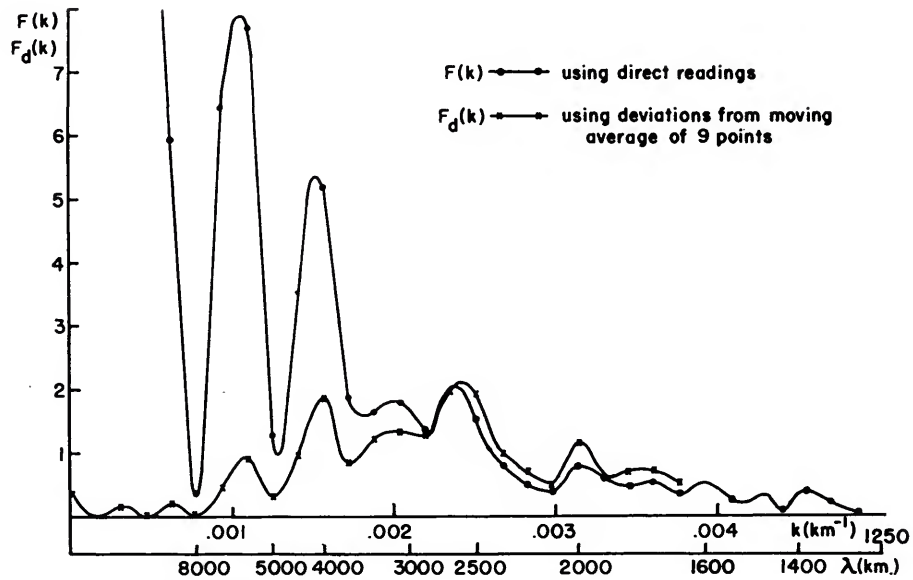


FIGURE 7.—Unidimensional frequency spectra derived from photospheric brightness data.

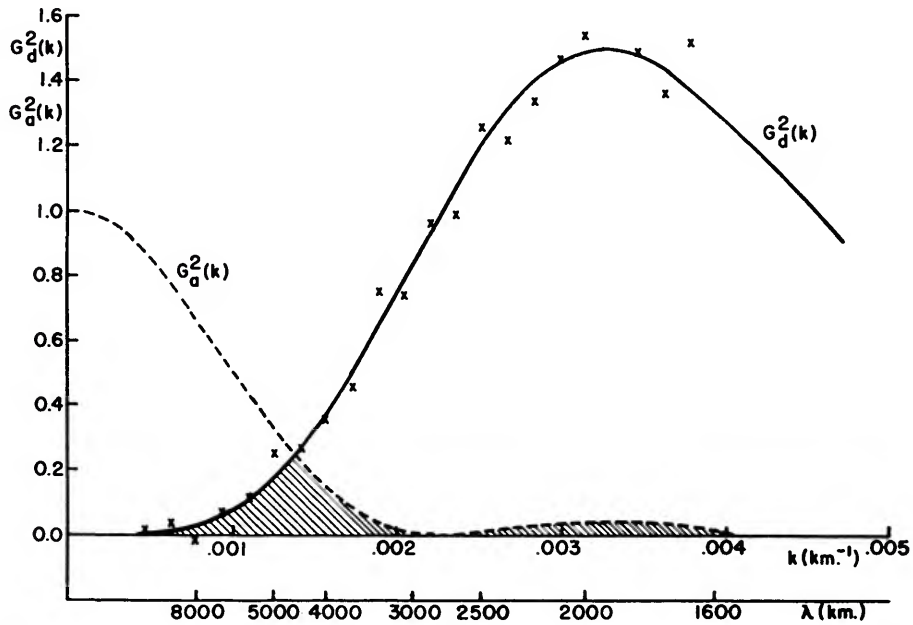


FIGURE 8.—Filtering of the frequency spectrum by the moving average procedure. Solid curve, filter $G_d^2(k)$ corresponding to the deviations from the moving average. Dotted curve, filter $G_a^2(k)$ corresponding to the moving average itself.

around a wavelength of about 2600 km and it is only weakly peaked. Thus the oscillations of small amplitude and of pseudoperiodicity 2500 km in figure 2 for $m > 8$, are to be expected.

When the moving average is defined in its simplest way by formula (4), it presents other disadvantages. The separation by formulae (4) and (5) between the long wavelength space fluctuations and the short wavelength space fluctuations is imperfect. Also the energies are very much altered. As a consequence, the variables Y_i and $A_i - \bar{T}_i$ are not statistically independent. The simplest way to prove this is to consider the spectrum $F_a(k)$ corresponding to $A_i - \bar{T}_i$, that is the moving average itself. It can be shown that, for a very large sample, the relation between $F_a(k)$ and the original spectrum $F(k)$ is given by:

$$F_a(k) = G_a^2(k) F(k) \quad (19)$$

where:

$$G_a k = \frac{2}{9} \left(\frac{1}{2} + \cos kD + \cos 2kD + \cos 3kD + \cos 4kD \right). \quad (20)$$

The dotted curve of figure 8 represents $G_a^2(k)$. The shaded area shows where the two filters overlap. As the overlapping takes place in a region where the spectrum $F(k)$ has a large amount of energy, the small and large wavelengths are not well separated.

On the other hand, it is a property of the Fourier transform that the energy of a process is proportional to the area under the curve representing its spectrum. For instance, in the discrete case corresponding to the variable X_i , the energy of the fluctuations is proportional to the covariance \bar{X}_i^2 and we have:

$$\bar{X}_i^2 = \frac{\pi}{65D} \sum_{h=0}^{65} \epsilon_h F(k), \epsilon_h \begin{cases} \frac{1}{2}, & h=0, 65 \\ 1, & \text{otherwise} \end{cases}. \quad (21)$$

The energy of the fluctuations would be conserved only if:

$$\sum_{h=0}^{65} [G_a^2(k) + G_d^2(k)] \epsilon_h F(k) = \sum_{h=0}^{65} \epsilon_h F(k). \quad (22)$$

There is no reason to have such an equality. In particular, in the case of the sample studied, I have computed:

$$\frac{\pi}{65D} \sum_{h=0}^{65} [G_a^2(k) + G_d^2(k)] \epsilon_h F(k) = 72.5 + 35.7 = 108.2$$

and

$$\frac{\pi}{65D} \sum_{h=0}^{65} \epsilon_h F(k) = 129.5.$$

Therefore for this sample:

$$\sum_{h=0}^{65} [G_a^2(k) + G_d^2(k)] \epsilon_h F(k) < \sum_{h=0}^{65} \epsilon_h F(k), \quad (23)$$

which can be written, in terms of covariances:

$$\overline{(A_i - \bar{T}_i)^2} + \bar{Y}_i^2 < \bar{X}_i^2. \quad (24)$$

As we have by definition

$$(A_i - \bar{T}_i) + Y_i = X_i, \quad (25)$$

the inequality (24) means that the variables $A_i - \bar{T}_i$ and Y_i are correlated. I have computed their crosscorrelation coefficient $C(o)$ and found:

$$C(o) = .20.$$

It would be completely erroneous to conclude from that result that there is a physical correlation between the long wavelength space fluctuations and the short wavelength ones.

Small scale structure on the sun

As the use of moving averages can give spurious results, the simplest way to examine whether microphotometric tracings of solar granulation reveal a small scale structure seems to be the direct computation of the frequency spectrum. In all cases where that has been done by me or by others (Frenkiel and Schwarzschild, 1955; Uberoi, 1955a), no significant narrow maximum has been found in the short wavelengths range ($< 3,000$ km). The work of Uberoi (1955b) seems to indicate that the result remains the same for the two-dimensional spectrum computed from the one-dimensional one by assuming a two-dimensional isotropy.

It has been pointed out by Frenkiel and Schwarzschild (1955) and by Uberoi (1955b) that, because of the diffraction by the objective, wavelengths smaller than 2π times the tele-scopic resolution are very attenuated or com-

pletely suppressed. This effect could hinder the discovery of a real maximum in the spectrum. Another way of explaining the absence of a narrow maximum appears in the following section.

Comparison of visual and photometric measurements

Visual measurements lead to a distribution of "granule" sizes presenting a narrow peak (Keenan, 1938; Macris, 1953; Miller, 1954). As this result apparently contradicts the microphotometer analysis, it is interesting to examine: a) the relation between the frequency distribution of granule diameters and the frequency spectrum derived from tracings, and b) the procedure used by visual observers.

Visual frequency distributions deal only with the bright regions on the photographs and concern only their sizes and not their respective positions. On the other hand, the spectral analysis is sensitive to periodicities and the spectrum takes into account the position of the granules as well as their shape. This is illustrated by the following example given by Schwarzschild (private communication): Let us consider the granules as sharp edged circles of precisely the same diameter L , the same brightness, randomly distributed on a uniform background. The visual frequency distribution is then very sharp while the two-dimensional frequency spectrum does not show any pronounced peak. It can be shown, moreover, that since the distribution of granules is random, the spectrum is the same as that of a single granule; this spectrum is nearly flat in the low frequency range (waves of length $\lambda > 2L$), and with increasing frequency it decreases rapidly with small oscillations. If, instead of being randomly spaced, the granules belonged to a quasiperiodic array, like the atoms of a crystal, then the spectrum would be peaked but the frequencies of the peaks would correspond not to the granule size but to the mean distance between them. As the actual frequency spectra derived from some granulation photographs do not show any

statistically significant peaks, we may conclude that, on these photographs, the granule distribution does not present any significant periodicity.

On the other hand, even though the absence of peaks in the spectra is well explained by the above example, the fact remains that the actual spectra differ from what we expect when we examine the frequency distributions of the visual observers. As these distributions are quite narrow we would expect, as in the example of Schwarzschild, that the spectrum will be quite flat in the low frequency range; instead, however, the spectra in that range appear to be of the low frequency type. That suggests we should analyze further the visual procedures. It appears that visual observers generally discard the large features ($>4''$ of arc) even if they are very conspicuous on the photographs.² Also, the observers generally attempt to characterize a given granule by a single parameter called its "diameter." In fact, the granules have an irregular shape and the observers must determine visually the diameter of the equivalent circle which would have the same area as the granule itself. By that procedure, they neglect the small scale structure present in the granule itself, especially at its edges. The narrow frequency distribution results essentially from this double selection.

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² Dr. W. A. Miller has studied these large features in detail but he has not included them in his frequency distribution of granule sizes.

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Abstract

Direct statistical analysis of microphotometer tracings of solar granulation photographs made by Dr. William A. Miller of Radio Corporation of America fails to reveal a periodic structure, and agrees with the conclusions of Uberoi and of Frenkiel and Schwarzschild. The previous conclusion of Stuart and Rush that there exists a narrow distribution of cell size was not justified by their analysis. It is shown that the procedure of the moving average used by Stuart and Rush leads to a pseudoperiodic autocorrelation function that has no physical meaning. Particularly, the pseudoperiod is nearly proportional to the range included in the moving average. The bases of visual and of photometric measurements are shown to be widely different. The narrow distribution of granule sizes found by visual observers is due to the simultaneous presence of at least two selection effects.