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GRANULATION AND OSCILLATIONS OF THE SOLAR ATMOSPHERE

by Charles Whitney

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Granulation and Oscillations of the Solar Atmosphere

By Charles Whitney [1]

The intensity of the continuous and line spectra of the solar disk and the velocities inferred from Doppler displacements show point-to-point fluctuations. The optical properties of the equipment and the earth's atmosphere greatly limit quantitative observations of these fluctuations.

These essentially random fluctuations may be described by three parameters (although improved observing techniques may show these averages to be meaningless): (1) the mean size, or characteristic length; (2) the intensity contrast; and (3) the mean lifetime or half-life.

Summary of observations

One-dimensional autocorrelation analyses indicate that the fluctuations in brightness are essentially random, although visual examination of photographs usually suggests the existence of characteristic lengths. In many cases the characteristic length thus derived (3000 to 5000 km) is undoubtedly a measure of the resolving power of the photograph.

Recent studies by Rösch (1955, 1957), however, support the view that a characteristic length of 1000 to 1500 km has a real significance. Indeed, he finds foreshortening toward the limb for granules of diameter 1000 km, which strongly suggests that these are not random clumpings of smaller granules.

Data concerning lifetime and contrast have been summarized by Macris (1953). Corrections for scattered light have led to estimates of intensity contrasts as high as 30 to 40 percent, values corresponding to fluctuations of about 500° K in brightness temperature. These estimates are extremely uncertain.

The lifetimes of granules are of the order of minutes, judging from the changing appearance of the granulation pattern. Quantitative estimates give 2 to 5 minutes as the mean half-life.

Evidence exists for a positive correlation between continuum brightness and violet shift of the Fraunhofer lines. On the basis of plots of velocity and brightness, Richardson and Schwarzschild (1950) suggest the presence of only a weak correlation. Plaskett's statistical analysis (1954) showed a correlation on two of the three plates he measured. However, the weakness of the correlations suggested to Plaskett that the absorption lines were formed above the granulation.

As will become evident in the final section of this paper, it is not possible to base a physical model on such correlations measured on isolated plates. The analysis of a time sequence, yielding phase relations, is required before meaningful statements can be made concerning the physical connection between fluctuations in velocity and brightness.

Visual examination of prints made from spectra obtained with the McMath-Hulbert vacuum spectrograph has convinced the present writer of a positive correlation between localized darkening of the photospheric continuum and the largest redward Doppler-displacements of metallic lines.

A fact which may place restrictions on theories of granulation is the striking similarity between the patterns of Doppler displacements of strong (chromospheric) Fraunhofer lines and of weaker lines, presumably formed in the photosphere. This similarity also holds be-

[1] Smithsonian Astrophysical Observatory, and Harvard College Observatory.

tween neutral and ionized lines. McMath, Mohler, Pierce, and Goldberg (1956) have concluded that this similarity "implies either a chromospheric origin for the centers of all medium-strong Fraunhofer lines or the extension of the photospheric granules into the low chromosphere."

Interpretations of the data

There is little doubt of a physical connection between granulation and the instability of the deep photosphere (optical depth greater than unity) against convective motions. But, is the connection direct or indirect? Early interpretations held that the connection was direct and that granules were the rising convective cells themselves. But this interpretation no longer seems so reasonable as it once did, and it has yet to be investigated quantitatively in terms of variations of opacity, excitation, and the relative depths of line and continuum formation. Since the temperature gradient of the upper photosphere is less than the adiabatic gradient, this region is stable against convective motions. Thus, it becomes difficult to explain the positive correlation between brightness and upward velocity or the extension of the motions into the low chromosphere.

Further, Plaskett's (1956) calculations imply that the persistence of granulation to within 5″ to 10″ of the solar limb (see, e. g., Rösch, 1955, 1957) is not consistent with the hypothesis that brightness fluctuations are generated as deep as the top of the convection zone (optical depth approximately unity).

If we admit the extension of granulation into the upper photosphere or low chromosphere, and recognize the stability of this region against convective motions, it becomes necessary to assume that granulation is an effect of wave motions.

That is, we must admit that the fluctuation energy is propagated into the critical region as wave energy, rather than being carried in as thermal energy by mass motions.

The suggestion that sound waves are present in the solar atmosphere is not a new one (Biermann, 1946; Schwarzschild, 1948; Schatzman, 1953; Thomas, 1954), but previous investigators have limited themselves to the one-dimensional case of plane-waves propagated vertically. However, I treat the two dimensional equations of motion and am led to consider a family of solutions which differ physically from those previously discussed. These solutions, which do not appear in the one-dimensional treatment, represent gravity waves or, more generally, mixtures of gravity waves and compressional waves.

I adopt an approach suggested to me by Krook in which a "top" of the convection zone is postulated and is treated as a solid surface whose Z-coordinate fluctuates with X, Y, and time. These fluctuations of Z may be considered as resulting from a combination of convective motions and sound waves. These motions of the "top" of the convective zone impress motions on the overlying atmosphere, and waves are generated. These waves will interfere constructively or destructively in a manner that depends on the characteristic frequencies and horizontal lengths of the fluctuations. Those that interfere constructively will be amplified, and can be expected to carry energy and momentum into the upper atmosphere.

The present paper analyzes this situation by (a) considering the problem in two space-dimensions, (b) assuming temperature and gravity to be independent of height in the initial, undisturbed atmosphere, and (c) by treating only the steady-state of motions set up by the Fourier components of the fluctuations of height at the "top" of the convection zone.

This restriction to steady-state solutions is a serious one, in view of the random nature of motions in the convection zone. We hope subsequently to remove this restriction and to treat the situation as an initial-value problem.

The equations of motion [2]

We shall use the Lagrangian formulation, and consider adiabatic oscillations in two space-dimensions. Capital letters and subscripts "0" shall denote unperturbed values, and lower-case letters shall denote deviations from equilibrium. The acceleration of gravity, g, is directed toward negative Z and is independent

[2] In this and the following section we use the development given by Bjerknes et al. (1934, chapters 7, 8).

of Z. Define α, β, γ by

$$\alpha=\frac{d\rho}{dp}=\frac{1}{a^2},\ \beta=\frac{d\rho_0}{dp},\ \gamma=\frac{c_p}{c_v},$$

where a can be shown to equal the velocity of propagation of a small disturbance in a homogeneous medium.

The equations of motion in their linearized form become:

$$\frac{\partial^2 x}{\partial t^2}+\frac{\partial}{\partial X}\frac{p}{\rho_0}+g\frac{\partial z}{\partial X}=0, \qquad (1)$$

$$\frac{\partial^2 z}{\partial t^2}+g\frac{\partial z}{\partial Z}+\frac{\partial}{\partial Z}\frac{p}{\rho_0}+(\alpha-\beta)\frac{p}{\rho_0}=0, \qquad (2)$$

$$\frac{\partial x}{\partial X}+\frac{\partial z}{\partial Z}+\gamma\frac{p}{\rho_0}=0. \qquad (3)$$

We assume solutions of the form

$$x=A(Z)\cos(kX-\omega t), \qquad (4)$$

$$z=C(Z)\sin(kX-\omega t), \qquad (5)$$

$$a^2\frac{\delta\rho}{\rho_0}=\frac{p}{\rho_0}=D(Z)\sin(kX-\omega t), \qquad (6)$$

and introduce them into equations (1) to (3), obtaining the equations,

$$-\omega^2 A+kgC+kD=0, \qquad (7)$$

$$-\omega^2 C+gC'+D'+g(\alpha-\beta)D=0, \qquad (8)$$

$$kA+C'+\alpha D=0. \qquad (9)$$

The primes denote differentiation with respect to Z.

Before discussing the most general solution to this set of equations, we shall investigate four special cases.

Case I: $A=0$, $k=0$.—In this case, x vanishes, and the phases of z and p are independent of X. The solution represents plane waves propagated vertically. Equation (9) is now

$$C'=-\alpha D$$

or

$$D'=-\frac{1}{\alpha}C''.$$

Introducing equation (10) into (8) leads to the equation,

$$C''-\beta gC'+\omega^2\alpha C=0. \qquad (11)$$

This equation has the solution

$$C(Z)=e^{\beta gZ/2}(C_1e^{\eta Z}+C_2e^{-\eta Z}), \qquad (12)$$

$$\eta^2=\frac{\beta^2g^2}{4}-\omega^2\alpha. \qquad (13)$$

When $\eta^2>0$, we have

$$z=e^{\beta gZ/2}(C_1e^{\eta Z}+C_2e^{-\eta Z})\sin\omega t, \qquad (14)$$

which represents the sum of two standing waves, each of which has an amplitude increasing upwards. The solution for p has the same Z-dependence.

When $\eta^2<0$, we write

$$\eta=i\left(\omega^2\alpha-\frac{\beta^2g^2}{4}\right)^{1/2}=i\zeta, \qquad (15)$$

and find

$$z=e^{\beta gZ/2}(C_1e^{i\zeta Z}+C_2e^{-i\zeta Z})\sin\omega t. \qquad (16)$$

This solution represents the combination of an upward-running wave and a downward-running wave. The phase velocity of these waves, V, is given by the expression

$$V=\frac{\omega}{\zeta}=\frac{a}{\left(1-\frac{\beta^2g^2a^2}{4\omega^2}\right)^{1/2}}>a. \qquad (17)$$

For the group velocity,

$$U=\frac{\partial\omega}{\partial\zeta},$$

we find from equation (15),

$$U=a\left(1-\frac{\beta^2g^2a^2}{4\omega^2}\right)^{1/2}<a. \qquad (18)$$

This is the velocity with which energy is transmitted through the medium.

For an atmosphere which is initially isothermal,

$$p=\rho_0\frac{kT}{\mu}=\rho_0\frac{a^2}{\gamma},$$

where k and μ are the Boltzmann constant and the mean molecular weight of the gas. Hence, since β is defined as

$$\beta=\gamma/a^2,$$

and, since $\alpha=1/a^2$, we have

$$\eta=\left(\frac{\gamma^2 g^2}{4a^4}-\frac{\omega^2}{a^2}\right)^{1/2}.$$

Therefore the value of ω separating the real and imaginary domains of η is

$$\omega_c=\frac{\gamma g}{2a}.$$

For the solar atmosphere, $g=10^{4.44}$ cm/sec^{-2}, $a=\sqrt{\gamma kT/\mu}$, $a=6.4\ 10^5\gamma^{1/2}$ cm/sec, $\mu\simeq m_H$, and $T\simeq 5000°$ K.

Hence

$$\tau_c=\frac{2\pi}{\omega_c}=\frac{3}{\sqrt{\gamma}}\ 10^2 \text{ sec},$$

and since $5/3>\gamma>1$, the critical period is between 3 and 5 minutes.

Case II: $A=0$, $k\neq 0$.—This is the case of purely transverse waves propagated horizontally, and we have from equations (7) to (9) the equation

$$C\,[\omega^2+g^2(\beta-\alpha)]=0.$$

Hence for a non-vanishing solution,

$$\omega^2=g^2(\beta-\alpha),$$

$$C'-\alpha gC=0, \qquad (19)$$

$$D=-gC. \qquad (20)$$

The solutions of equations (19) and (20) are

$$C(Z)=C_0e^{\alpha gZ},$$

$$D(Z)=-gC_0e^{\alpha gZ}.$$

We note that the frequency of these oscillations is independent of k, the wave number, and that it vanishes for an atmosphere in which $\alpha=\beta$.

Two types of atmospheres for which $\alpha=\beta$ are *a*) an atmosphere in adiabatic equilibrium, and *b*), an atmosphere which is initially isothermal and which undergoes isothermal changes of state ($\gamma=1$).

For the solar conditions adopted above, we have

$$\omega_t^2=\frac{g^2}{a^2}(\gamma-1)=\frac{\gamma-1}{\gamma}\ 1.84\ 10^{-3},$$

$$\tau_t=\frac{2\pi}{\omega_t}=1.5\ 10^2\left(\frac{\gamma}{\gamma-1}\right)^{1/2}\text{sec}.$$

When $\gamma=5/3$, $\tau_t=2.3\ 10^2$ sec, which is nearly identical to the critical period for plane, vertical waves. In fact,

$$\frac{\omega_c}{\omega_t}=\frac{\gamma}{2\sqrt{\gamma-1}}.$$

Case III: $C=0$.—Equations (7) to (9) give

$$\omega^2=k^2a^2,$$

$$D'+g(\alpha-\beta)D=0,$$

$$A=\frac{1}{ka^2}D.$$

Hence

$$D(Z)=D_0e^{g(\beta-\alpha)Z},$$

$$A(Z)=\frac{1}{ka^2}e^{g(\beta-\alpha)Z}.$$

This solution represents plane longitudinal waves running horizontally. These are pure sound waves. The amplitude of the wave shows an exponential increase with height when $\beta>\alpha$, i. e., when

$$\frac{d\rho_0}{dp}>\frac{d\rho}{dp}.$$

The case $\alpha=\beta$ is distinguished by an amplitude independent of height.

Case IV: $D=0$.—The pressure variations now vanish and we have pure gravity waves. That is, the potential energy of wave motion appears in the form of gravitational potential alone. This is to be contrasted with the solution in Case III, in which the gravitational potential of a particle is independent of time (since $z=0$), and the wave potential has the form of compression energy. Equations (7) to (9) give the expressions

$$\omega=\sqrt{kg},$$

$$x=A_0e^{kZ}\cos(kX-\omega t),$$

$$z=A_0e^{kZ}\sin(kX-\omega t).$$

The wave amplitude increases with height, and the particle paths are circular. Since α and β do not appear in this solution, and since the pressure fluctuations vanish everywhere, this solution is also valid for an infinitely deep, homogeneous, incompressible fluid with a free surface.

Summary of particular solutions.—We summarize the results of the above discussion by assembling the dispersion relations for the solutions considered. In the parenthetical comments, the words "vertical" and "horizontal" refer to the direction of phase propagation, or, equivalently, the direction of motion of the planes of constant phase. The words "transversal" and "longitudinal" refer to waves in which the particle motions are, respectively, perpendicular to and parallel to the direction of phase propagation. The absence of vertical-transverse waves is due to the atmosphere's inability to resist shearing motion, i. e., the neglect of viscosity.

Case I: $\omega_c=\frac{\gamma g}{2a}$,	(Critical frequency for vertical-longitudinal wave.)
Case II: $\omega_t=(\gamma-1)g/a$	(Horizontal - transverse wave.)
Case III: $\omega_s=ka$,	(Horizontal - longitudinal, or pure sound wave.)
Case IV: $\omega_g=\sqrt{kg}$.	(Pure gravity wave. Phase propagated horizontally. Circular particle paths.)

Solution for an isothermal atmosphere

If we eliminate C and D from equations (7) to (9), and set α and β constant, we find that

$$A''-g\beta A'+\left[\alpha\omega^2-k^2+g(\beta-\alpha)\frac{k^2}{\omega^2}\right]A=0. \quad (21)$$

C and D satisfy identical equations when α and β are constant.

From equations (7) to (9) we may also find that

$$C=\frac{\omega^2}{k}\frac{A'-g(\beta-\alpha)A}{\omega^2-g^2(\beta-\alpha)}, \quad (22)$$

$$D=\frac{\omega^2}{k}\frac{-gA'+\omega^2A}{\omega^2-g^2(\beta-\alpha)}. \quad (23)$$

Thus, given the solution to equation (21), the values of C and D may be found from equations (22) and (23). The results are:

$$A=e^{g\beta Z/2}[A_1e^{\eta Z}+A_2e^{-\eta Z}], \quad (24)$$

$$C=\frac{\omega^2}{k}\frac{e^{g\beta Z/2}}{\omega^2-g^2(\beta-\alpha)}\left\{\left[g\left(\alpha-\frac{1}{2}\beta\right)+\eta\right]A_1e^{\eta Z}+\left[g\left(\alpha-\frac{1}{2}\beta\right)-\eta\right]A_2e^{-\eta Z}\right\}, \quad (25)$$

$$D=g\frac{\omega^2}{k}\frac{e^{g\beta Z/2}}{\omega^2-g^2(\beta-\alpha)}\left\{\left[g\left(\frac{\omega^2}{g^2}-\frac{1}{2}\beta\right)-\eta\right]A_1e^{\eta Z}+\left[g\left(\frac{\omega^2}{g^2}-\frac{1}{2}\beta\right)+\eta\right]A_2e^{-\eta Z}\right\}. \quad (26)$$

We now impose the conditions that the atmosphere be isothermal in space and time ($\alpha=\beta=1/a^2$). Introducing the scale height, H, whose value is

$$H=a^2/g,$$

we find

$$A=e^{Z/2H}(A_1e^{\eta Z}+A_2e^{-\eta Z}), \quad (27)$$

$$C=\frac{1}{k}e^{Z/2H}\left[\left(\frac{1}{2H}+\eta\right)A_1e^{\eta Z}+\left(\frac{1}{2H}-\eta\right)A_2e^{-\eta Z}\right], \quad (28)$$

$$D=\frac{1}{k}e^{Z/2H}\left\{\left[\omega^2-g\left(\frac{1}{2H}+\eta\right)\right]A_1e^{\eta Z}+\left[\omega^2-g\left(\frac{1}{2H}-\eta\right)\right]A_2e^{-\eta Z}\right\}, \quad (29)$$

where

$$\eta=\sqrt{\frac{1}{4H^2}+k^2-\frac{\omega^2}{a^2}}.$$

Application of boundary conditions

If we consider the wave motions of the solar atmosphere to be driven by the upward and downward movement of the "top" of the convective zone, we may postulate a lower-boundary condition of the form

Condition I: $x(0)=0$,
Condition II: $z(0)=A_0H\sin kX\sin\omega t$.

These conditions state that the motions at the lower boundary of the atmosphere ($Z=0$) are vertical and are represented by a standing wave whose wave number is k.

The form of the solution initially assumed in equations (4) and (5) is that of a running wave. If, now, we assume a solution of the form

$$x=A(Z)\cos kX\sin\omega t, \quad (30)$$

$$z=C(Z)\sin kX\sin\omega t, \quad (31)$$

$$a^2\frac{\delta\rho}{\rho_0}=\frac{p}{\rho_0}=D(Z)\sin kX\sin\omega t, \quad (32)$$

the equations for the coefficients A, C, D are unchanged. Since the boundary condition has the form of a standing wave, it is clear that we should adopt equations (30) to (32) for further discussion.

The boundary conditions give

Condition I:

$$A(0)=0.$$

$$A_1=-A_2.$$

Condition II:

$$\left(\frac{1}{2H}+\eta\right)A_1+\left(\frac{1}{2H}-\eta\right)A_2=A_0kH.$$

From these we find

$$A_1=A_0kH/2.$$

Introduction of these into equations (27) to (29) gives

$$A(Z)=\frac{A_0kH}{2\eta}e^{Z/2H}(e^{\eta Z}-e^{-\eta Z}), \tag{33}$$

$$C(Z)=\frac{A_0H}{2}e^{Z/2H}\left[\left(\frac{1}{2H\eta}+1\right)e^{\eta Z}-\left(\frac{1}{2H\eta}-1\right)e^{-\eta Z}\right], \tag{34}$$

$$D(Z)=\frac{A_0H}{2}e^{Z/2H}\left\{\left[\frac{\omega^2}{\eta}-g\left(\frac{1}{2H\eta}+1\right)\right]e^{\eta Z}-\left[\frac{\omega^2}{\eta}-g\left(\frac{1}{2H\eta}-1\right)\right]e^{-\eta Z}\right\}. \tag{35}$$

Solutions (33) to (35) are completely specified by the parameters k and ω or, equivalently, k and η. There is a double infinity of possible solutions corresponding to the infinite range of choices for ω and k.

We now introduce a further condition on the solution, that the pressure variations vanish at a prescribed height, Z_t. Such a condition is difficult to justify fully, but its reasonableness is suggested by the following. First, the recent models indicate that the chromosphere-corona transition consists in an abrupt rise in temperature. For waves of sufficient length, this transition will appear as a discontinuity of temperature and density. The density above the discontinuity will be significantly less than that below the discontinuity, since the ratio of densities is essentially the reciprocal of the ratio of temperatures. Thus, for a sufficiently rapid and great increase of temperature with height, the discontinuity will behave essentially as a free (constant-pressure) boundary.

The second consideration is that dissipation of wave energy by radiation (excluded from the present solution, since $\gamma=1$) may damp the wave and limit its amplitude at great heights.

The condition that pressure variations shall vanish at height Z_t gives the relation:

$$D(Z_t)=0,$$

or

$$\left[\frac{\omega^2}{\eta}-g\left(\frac{1}{2H\eta}+1\right)\right]e^{2\eta Z_t}=\frac{\omega^2}{\eta}-g\left(\frac{1}{2H\eta}-1\right).$$

Hence

$$Z_t=\frac{1}{2\eta}\log_e\frac{\omega^2-g\left(\frac{1}{2H}-\eta\right)}{\omega^2-g\left(\frac{1}{2H}+\eta\right)}. \tag{36}$$

Thus, once the physical parameters Z_t, g, and H are given, equation (36) becomes a "dispersion" relation between ω and k.

When $\eta^2<0$, it is convenient to introduce, again,

$$i\zeta=\eta,$$

in which case the lower-boundary conditions, I and II, give

$$A_1=-\frac{iA_0kH}{2\zeta}.$$

The condition at the upper boundary becomes

$$\frac{k}{2i\zeta}\left[\omega^2-g\left(\frac{1}{2H}+i\zeta\right)\right]e^{i\zeta Z_t}-\frac{k}{2i\zeta}\left[\omega^2-g\left(\frac{1}{2H}-i\zeta\right)\right]e^{-i\zeta Z_t}=0. \tag{37}$$

This may be transformed to

$$\zeta Z_t-\tan^{-1}\left[\frac{1}{g\zeta}\left(\frac{g}{2H}-\omega^2\right)\right]=(n+1/2)\pi,\ (n=1,2,3\ldots), \tag{38}$$

or, finally,

$$\omega^2=\frac{g}{H}\left(\frac{1}{2}+\zeta H\cot\zeta H\frac{Z_t}{H}\right). \qquad (39)$$

Solutions (33) to (35) become

$$A(Z)=\frac{A_0kH}{\zeta}e^{Z/2H}\sin\zeta Z, \qquad (40)$$

$$C(Z)=$$
$$A_0H\left(1+\frac{1}{4H^2\zeta^2}\right)^{1/2}e^{Z/2H}\cos\left(\zeta Z-\tan^{-1}\frac{1}{2H\zeta}\right),$$

$$D(Z)=A_0Hg\sqrt{1+\frac{1}{H^2\zeta^2}\left(\frac{1}{2}-\omega^2\frac{H}{g}\right)^2}\,e^{Z/2H}$$
$$\cos\left[\zeta Z-\tan^{-1}\left(\frac{1}{2H\zeta}-\frac{\omega^2}{g\zeta}\right)\right]. \qquad (41)$$

Equation (41) reduces to

$$D(Z)=-A_0Hg\frac{\sin\zeta(Z-Z_t)}{\sin\zeta Z_t} \qquad (42)$$

In the special case $\eta^2=0$, equation (21) and the lower-boundary condition give

$$A(Z)=A_0HkZ\,e^{Z/2H}, \qquad (43)$$

$$C(Z)=A_0H\left(1+\frac{Z}{2H}\right)e^{Z/2H}, \qquad (44)$$

$$D(Z)=A_0HZ\left(\omega^2-\frac{g}{2H}-\frac{g}{Z}\right)e^{Z/2H}. \qquad (45)$$

The condition of vanishing pressure at height Z_t gives

$$Z_t\left(\omega^2-\frac{g}{2H}\right)-g=0,$$

hence

$$k^2H^2=\frac{H}{Z_t}+\frac{1}{4}. \qquad (46)$$

Evaluation of dispersion relation ($\eta^2\geq0$).—We shall now evaluate these dispersion relations, commencing with the case $\eta^2=0$. From the assumed physical conditions, and taking $Z_t=10H$, we derive

$$\frac{2\pi}{k}=1600\text{ km},$$

$$\frac{2\pi}{\omega}=189\text{ sec}.$$

These results are not sensitive to Z_t as long as $Z_t\geq 10\,H$. When $\eta^2>0$, equation (36) becomes

$$20\eta H=\log_e\frac{\omega^2+1.84\;10^{-3}\eta H-0.92\;10^{-3}}{\omega^2-1.84\;10^{-3}\eta H-0.92\;10^{-3}}. \qquad (47)$$

For each assumed value of ω, solution of equation (47) gives the value of ηH. From the definition of η, we then derive

$$k^2H^2=\eta^2H^2+\omega^2\frac{H}{g}-\frac{1}{4}. \qquad (48)$$

It is easily shown that equation (47) has no non-zero, real solution for η when $\omega^2\leq 1.10\;10^{-3}$. Also, in the limit

$$\omega^2\to 1.10\;10^{-3}$$

we have

$$\eta H\to 0,$$

and the solution approaches that found above for $\eta=0$. Therefore, we conclude that when $\eta>0$ the period of oscillation and the "wavelength" (λ) in the horizontal direction must satisfy, respectively, the conditions

$$T^2\leq\frac{4\pi^2}{1.10\;10^{-3}}\text{ sec}^2,$$

$$\lambda^2\leq\frac{4\pi^2H^2}{0.35}.$$

That is, the period must be less than 190 sec and the wavelength less than 1600 km. When we take $Z_t=\infty$, the dispersion relation is changed only slightly, and the limiting period and wavelength are 208 and 1800 km, respectively.

Tables 1 and 2 list the solutions for $Z_t=10H$, and $Z_t=\infty$.

When $\frac{2\pi}{\omega}\leq 140$ sec, the dispersion relation becomes

$$\omega^2=kg,$$

which is the relation for pure gravity waves. Consider, for example, a solution corresponding to $Z_t=10H$, $\omega^2=2.00\;10^{-3}$, $A_0=10^{-5}$. The vertical displacements at the lower boundary have an amplitude $A_0H=150$ cm. The horizontal and vertical displacements at $Z=10H$ have an amplitude, according to table 1 of

$$A=C=4.9\;10^4\;A_0H=74\text{ km}.$$

TABLE 1.—*Solutions of dispersion relation and corresponding amplitudes*

($\eta^2 \geq 0$, $Z_t = 10\,H$)

ω^2	Period (sec) $\frac{2\pi}{\omega}$	Wave-length (km) $\frac{2\pi}{k}$	Displacement amplitudes $\frac{A(10\,H)}{A_0 H}$	$\frac{C(10\,H)}{A_0 H}$	Relative density variation $A_0^{-1}\left(\frac{\delta\rho}{\rho_0}\right)_{z=5\,H}$	Product of wave number and gravity (kg)
1.10×10^{-3}	189	1600	8.8×10^{2}	8.9×10^{2}	-5.9	1.07×10^{-3}
1.20×10^{-3}	181	1450	1.3×10^{3}	1.3×10^{3}	-6.3	1.17×10^{-3}
2.00×10^{-3}	140	868	4.9×10^{4}	4.9×10^{4}	-6.6×10^{-1}	1.98×10^{-3}
3.00×10^{-3}	115	580	7.3×10^{6}	7.3×10^{6}	-4.4×10^{-2}	3.00×10^{-3}
3.95×10^{-3}	100	440	1.3×10^{9}	1.3×10^{9}	-3.4×10^{-3}	3.95×10^{-3}
3.95×10^{-2}	32	44	$4\ \times10^{92}$	$4\ \times10^{92}$	-5.4×10^{-45}	3.95×10^{-2}

TABLE 2.—*Solution of dispersion relation and corresponding amplitudes*

($\eta^2 \geq 0$, $Z_t = \infty$)

ω^2	Period (sec) $\frac{2\pi}{\omega}$	Wave-length (km) $\frac{2\pi}{k}$	Displacement amplitudes $\frac{A(10\,H)}{A_0 H}$	$\frac{C(10\,H)}{A_0 H}$	Relative density variation $A_0^{-1}\left(\frac{\delta\rho}{\rho_0}\right)_{z=5\,H}$	Product of wave number and gravity (kg)
0.92×10^{-3}	208	1800	7.4×10^{2}	9.0×10^{2}	-1.2×10^{1}	0.92×10^{-3}
1.10×10^{-3}	189	1510	1.0×10^{3}	1.1×10^{3}	-7.6	1.10×10^{-3}
1.20×10^{-3}	181	1380	1.4×10^{3}	1.4×10^{3}	-5.7	1.20×10^{-3}
2.00×10^{-3}	140	830	4.7×10^{4}	4.7×10^{4}	-6.6×10^{-1}	2.00×10^{-3}
3.00×10^{-3}	115	550	8.2×10^{6}	8.2×10^{6}	-4.4×10^{-2}	3.00×10^{-3}
3.95×10^{-3}	100	420	1.3×10^{9}	1.3×10^{9}	-3.4×10^{-3}	3.95×10^{-3}
3.95×10^{-2}	32	4	$4\ \times10^{92}$	$4\ \times10^{92}$	-5.4×10^{-45}	3.95×10^{-2}

The density and pressure variations vanish at $Z=10H$, but at the intermediate height $Z=5H$, table 1 gives an amplitude

$$\frac{\delta\rho}{\rho_0} = -0.66\,A_0 = -6.6\,10^{-6}.$$

The pressure and density variations are, indeed, small.

The horizontal and vertical components of particle velocity are

$$\dot{x} = A\omega \cos kX \cos \omega t,$$
$$\dot{y} = C\omega \sin kX \cos \omega t.$$

Hence, the amplitudes of the velocity variations at $Z=10H$ are $A=C=2.8$ km/sec$=0.44a$, while the amplitudes of the gradients of displacement are $Ak=Ck=0.53$. These amplitudes are not much smaller than unity, and the linearized equations of motion are not valid.

If we consider a solution whose amplitude at $Z=0$ is about one-fifth the amplitude used above, i. e., take

$$A_0 = \frac{1}{4.9}\,10^{-5},$$

the linear equation will be nearly valid. In this case, the solution for the motion at the height $Z=10H$ is

$$x = \frac{H}{10} \cos kX \sin \omega t,$$

$$y = \frac{H}{10} \sin kX \sin \omega t,$$

The X and Z motions are in phase in time, but in space they have a quarter-wave difference in phase. Therefore, the particle paths are rectilinear, varying from vertical, when $kX=(2n+1)\frac{\pi}{2}$, to horizontal, when $kX=n\pi$.

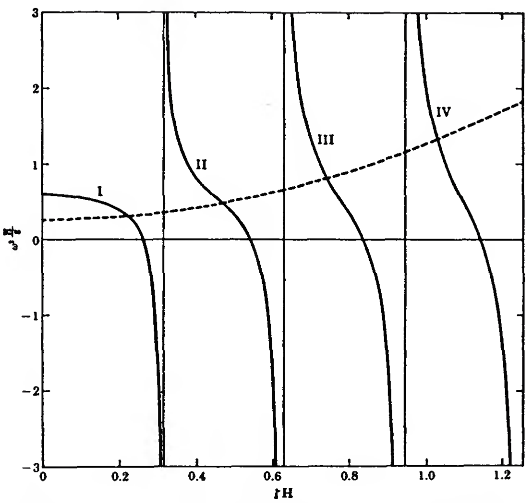

FIGURE 1.—Solution of dispersion relation for an isothermal atmosphere driven from below by an oscillating corrugation ($\eta^2<0$). Abscissa: ζH, where ζ is defined by equation (49). Ordinate: values of $\omega^2 \frac{H}{g}$ derived from equations (39) and (49). The dotted line denotes the lower limit to values of the ordinate allowed by the condition $\eta^2<0$. Roman numerals denote consecutive branches of the solution.

The motions converge horizontally toward the regions of upward displacement and diverge from the regions of downward displacement. The resulting wave form is not sinusoidal, and it deviates in the sense that the crests are "sharper" than the troughs; that is, the crests have a smaller radius of curvature than the troughs. As long as the linear solution is valid, this peaking is not very marked, and the wave form is nearly sinusoidal. The peaking is enhanced in the nonlinear case of finite amplitude.

Evaluation of dispersion relation ($\eta^2<0$).—When $\eta^2<0$, we have

$$\zeta^2=-\eta^2=\frac{\omega^2}{a^2}-k^2-\frac{1}{4H^2}>0 \tag{49}$$

and we employ the dispersion relation (39). The solution of equation (39), with $Z_t=10H$, is shown in figure 1, where $\omega^2\frac{H}{g}$ is plotted against ζH (solid lines). The condition on ζ^2 expressed by equations (49) gives

$$\omega^2\frac{H}{g}>\zeta^2H^2+\frac{1}{4}. \tag{50}$$

The dotted line in figure 1 is the curve for which

$$\omega^2\frac{H}{g}=\zeta^2H^2+\frac{1}{4}. \tag{51}$$

Allowed solutions fall above this line. Thus, for example, when $\omega^2\frac{H}{g}=1$, there are only two allowed values of ζH, corresponding to the second and third branches of the solution, respectively.

The lowest allowed frequency on each branch of the solution follows from simultaneous solution of equations (39) and (51). In each case

$$2\pi/k=\infty,$$

hence the oscillation has the form of a plane wave. We find, for the first four branches,

$$\frac{2\pi}{\omega_{01}}=264 \text{ sec}, \qquad \frac{2\pi}{\omega_{03}}=163,$$

$$\frac{2\pi}{\omega_{02}}=212, \qquad \frac{2\pi}{\omega_{04}}=128.$$

As frequency increases above ω_{01}, the solution follows Branch I, until the frequency ω_{02} is attained. Beyond this frequency, solutions on Branch II are also allowed. For still higher frequencies, solutions on the succeeding branches become allowed. For large frequencies the number of allowed solutions becomes proportional to the frequency.

Figure 2 is a dispersion curve in which the logarithms of $2\pi/k$ and $2\pi/\omega$ are the coordinates. Only the first four branches are included. The remaining branches would appear at equal intervals to the left of those depicted. Note that, for a given frequency, the interval between allowed wavelengths increases with wavelength. That is, the shorter wavelengths are crowded together. Also, for periods less than, say, 10^2 seconds, the crowding is such that most of the allowed wavelengths lie near the straight asymptote. The equation of this asymptote is

$$k^2H^2=\omega^2\frac{H}{g},$$

or

$$\frac{2\pi}{\omega}a=\frac{2\pi}{k}=\lambda_H. \tag{52}$$

That is, the relation between the period and the horizontal wavelength, λ_H, is just the rela-

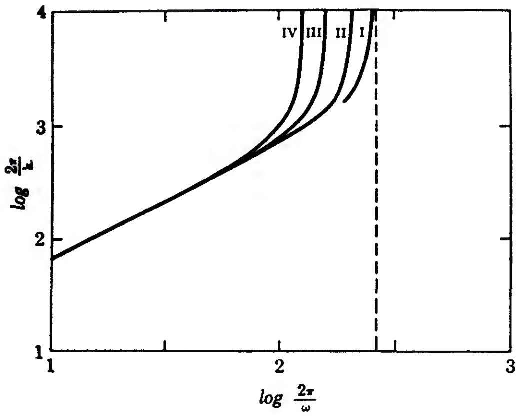

FIGURE 2.—Solution of dispersion relation ($\eta^2<0$). Abscissa: wavelength of driving corrugation, $2\pi/k$. Ordinate: period of oscillation, $2\pi/\omega$. Dotted line shows the upper limit to the period.

tion between period and wavelength of an acoustic wave.

Evaluation of equations (40) to (42) shows that the amplitudes of oscillation increase exponentially upward with an e-folding distance of two scale-heights. These increases are modulated by a circular function of height, and thus show nodes. The first branch shows no nodes but each succeeding branch has one additional node. Further, the horizontal and vertical amplitudes are comparable to each other, except for the cases $kH\rightarrow 0$ and $kH\rightarrow\infty$.

Comparison of the density oscillations for this type of solution ($\eta^2<0$) shows them comparable to those of the previous type ($\eta^2>0$) for periods near the upper limit. However, the density oscillations are nearly independent of period for this type, and hence become greater than those of the previous type as the period becomes shorter.

We may designate solutions for $\eta^2<0$ and $\eta^2>0$ as compressional and gravitational oscillations, respectively. This dichotomy is used only to indicate that the potential energy associated with the oscillations appears primarily as compressional energy in one case, and as work done against gravity in the other.

The difference in character of the compressional and gravitational solutions may be summarized as follows. The compressional solutions show an amplitude increasing as exp ($Z/2H$) and modulated by sine functions of height and horizontal distance. The density oscillations are virtually independent of periods. The gravitational solutions show a much more rapid increase with height, but they do not show a vertical modulation. The density oscillations decrease rapidly with period.

Both types of solutions are restricted to periods less than about four minutes, and the gravitational solution is restricted to horizontal wavelengths less than about 2000 km.

Thermal properties of the oscillations

To evaluate amplitudes, we took $\gamma=1$, recognizing that radiative exchange of energy will tend to make the oscillations isothermal. We may obtain an estimate of the amplitude and phase of temperature changes in the following way (Whitney, 1955).

Consider an optically thin element of matter whose equilibrium temperature and density are T_0 and ρ_0. Let the instantaneous values of these parameters be designated

$$T=T_0+\delta T,$$

$$\rho=\rho_0+\delta\rho,$$

and use similar notation for the other variables. Then the conservation of energy requires

$$\frac{R\rho_0}{\gamma-1}\frac{\partial T}{\partial t}-RT_0\frac{\partial\rho}{\partial t}=\kappa\rho\bar{I}=\rho\epsilon \qquad (53)$$

where κ, $\bar{I}$ and ϵ are respectively the absorption coefficient, the mean intensity of incident radiation, and the rate of emission. If

$$\bar{I}_0=\epsilon_0/\kappa_0,$$

and

$$\delta(\epsilon/\kappa)=4\sigma T_0^3\delta T,$$

it follows that

$$\frac{R\rho_0T_0}{\gamma-1}\frac{\partial}{\partial t}\frac{\delta T}{T_0}+4\sigma\kappa_0\rho_0T_0^4\frac{\delta T}{T_0}=R\rho_0T_0\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0}. \qquad (54)$$

The assumption of sinusoidal variations of $\delta\rho$ and δT, i. e.,

$$\frac{\delta T}{T_0}=Ee^{i\omega t},$$

$$\frac{\delta\rho}{\rho_0}=Fe^{i\omega t},$$

leads to the equation,

$$E = \frac{F(\gamma-1)}{1 - i\frac{4\kappa_0\sigma\rho_0 T_0^4}{\omega}\left(\frac{R\rho_0 T_0}{\gamma-1}\right)^{-1}}. \quad (55)$$

Defining

$$\vartheta = 4\kappa_0\rho_0\sigma T_0^4\left(\frac{R\rho_0 T_0}{\gamma-1}\right)^{-1},$$

we find

$$\frac{E}{F} = \frac{\gamma-1}{(1+\vartheta^2/\omega^2)^{1/2}} e^{i\tan^{-1}(\vartheta/\omega)}. \quad (56)$$

Hence, for vanishing opacity, $\vartheta=0$, and the temperature and density variations are in phase with each other.

Adopting the values of opacity and temperature in the model solar atmosphere tabulated by Minnaert (1953) we derive the amplitudes and phases given in tables 3 and 4.

These tables indicate that, for the optical depths contributing to continuum formation ($\tau > 0.1$) and for periods greater than 10 seconds, the radiative exchange of energy is significant. For a given density oscillation, the temperature oscillation is reduced and advanced in phase. Thus, maximum temperature can occur nearly at the phase of maximum rate of compression rather than at maximum compression itself.

TABLE 3.—*Values of* $(1+\vartheta^2/\omega^2{}_2)^{-1/2}$

Optical depth (τ)	Period of oscillation $\left(\frac{2\pi}{\omega}\right)$ in seconds			
	1	10	10^2	10^3
0. 0001	1. 00	1. 00	1. 00	0. 91
. 001	1. 00	1. 00	0. 93	. 24
. 129	1. 00	0. 99	. 55	. 07
. 294	1. 00	. 97	. 37	. 04
. 672	1. 00	. 89	. 19	. 02

If we consider 100 seconds, or greater, as the order of magnitude of observed granule lifetimes, we see from table 3 that the temperature oscillations, for a given density oscillation, increase rapidly with height in the atmosphere. Further, since the wave solutions show a density oscillation increasing with height, the temperature oscillations associated with such a wave must increase with height.

These arguments suggest that, if we associate the observed granulation with these waves, the mean height of granule formation must be greater than that of the mean continuum.

Concerning the phase relations between velocity and temperature, we note that the wave solutions show density oscillation and displacement to be 180° out of phase. Hence, maximum downward velocity is in phase with maximum rate of compression. For adiabatic oscillations ($\vartheta/\omega \ll 1$) maximum temperature occurs at maximum downward displacement. For oscillations significantly affected by radiative exchange, ($\vartheta/\omega \gtrsim 1$) the temperature phase-leads given in table 4 tend to shift maximum temperature toward the time of maximum downward velocity.

TABLE 4.—*Values of* $\tan^{-1}(\vartheta/\omega)$ *in degrees.*

Optical depth (τ)	Period of oscillation $\left(\frac{2\pi}{\omega}\right)$ in seconds			
	1	10	10^2	10^3
0. 0001	0. 026	0. 26	2. 6	24. 1
. 011	. 23	2. 3	22. 3	76. 2
. 129	. 86	8. 6	56. 1	85. 9
. 294	1. 4	14. 3	68. 2	87. 6
. 672	2. 9	26. 9	79. 1	88. 8

In any physical situation the phase relations will be intermediate between these extremes, and the maximum temperature will occur between the times of maximum and zero velocity downward. Thus these wave solutions give a negative correlation between upward velocity and temperature perturbation. Spectra of the solar disk in which granulation is discernible seem to indicate a positive correlation (see p. 366).

However, a positive correlation would arise from the present theory if the wave solution had the character of a running wave rather than the standing wave derived here. A running-wave solution would have resulted if we had not imposed the condition of vanishing pressure-oscillation at a specified height in the atmosphere. Whereas maximum density corresponds in phase to maximum displacement for the standing wave, it corresponds to maximum upward velocity for the running wave. Thus, application of the above arguments to a running-wave solution suggests that the maximum temperature should occur between zero velocity and maximum velocity upward, depending on the value of ϑ/ω.

Thus we may draw the following distinction: For a running wave, the maximum temperature should fall at or before maximum velocity upward; for the standing wave it should fall at or after maximum velocity downward.

To use this distinction as a criterion for choosing between standing and running waves as the cause of the observed granulation requires the assumption that there is no change of wave phase with height in the relevant region of the solar atmosphere. This assumption is not inconsistent with results of high-resolution spectroscopy discussed on page 365.

Admitting then, that the positive correlation between upward velocity and temperature rules out the standing-wave solution, we adopt the other alternative. But for the running solution, there is no unique dispersion-relation between frequency and horizontal wavelength—each value of one allows all values of the other.

Thus we can say nothing about the spectrum of horizontal wavelengths. Tables 3 and 4, however, allow the comment that since optical depths less than 0.1 contribute only trivially to the continuous radiation, the lifetimes of significant temperature fluctuations must be less than 10^3 seconds.

Also, the earlier arguments leading to a height of granule formation greater than that of the undisturbed continuum are still valid, since the running-wave solution also displays an amplification with height.

It is a pleasure to acknowledge the discussions with Max Krook which led to this study.

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Abstract

The observations of solar granulation are briefly summarized and their interpretation is discussed. Steady-state solutions of the linearized equations of motion in two dimensions, subject to the boundary condition of an oscillating corrugation at the bottom of the solar atmosphere, are obtained and their observable properties outlined.

Two points are emphasized. First, the motions of the solar atmosphere cannot be pure compression-waves, even in the region stable against convection. They should be considered as a mixture of compressional and gravitational waves. Second, a physical interpretation of granulation requires the determination, with time-resolved spectra, of the phase relations between brightness and velocity.

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