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GEODETIC USES OF ARTIFICIAL SATELLITES

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Geodetic Uses of Artificial Satellites

By George Veis [2]

Introduction

The geodetic methods heretofore applied to the data provided by the artificial satellites have been based on a dynamic approach. That is, the perturbations in the motion of a satellite have been used to determine the gravitational field of the earth, and thus to obtain information on the shape of the geoid (Jacchia, 1958b; O'Keefe, 1958; O'Keefe and Eckels, 1958; Kozai, 1959).

This paper presents another way of using the satellites in geodesy. The methods depend on a geometric approach, which we may say is related to mathematical geodesy in the same way that the dynamic approach is related to physical geodesy. The geometric method consists in performing a triangulation in space and determining the positions of a certain number of observing stations whose positions are unknown.

The science of geodesy today covers a wide field, but its main purpose is to determine the size and shape of the earth as a whole, or of large areas on its surface. This problem, at least at first sight, may seem to be a purely geometric one. But as soon as geodesists began using the direction of the vertical as a reference direction, the problem became dynamic as well. Selecting the vertical as the direction of reference is fully justified for the survey of a small area—say, a few square kilometers—since we can assume that the directions of the vertical are parallel to themselves over that area. But to relate them over large areas at different points on the earth we need to know the shape of the geoid, since the vertical is defined as being perpendicular to the geoid.

Since an approximation for the geoid is an ellipsoid of revolution, the geoid over a large area can be replaced by a reference ellipsoid. We therefore use the vertical as if it were the normal to the ellipsoid, and thus establish the relation between the reference directions. This is the method of "classical" geodesy.

The angle between the normal and the vertical (deflection of the vertical) can attain several seconds of arc, however, and over very large areas (of the order of continents) the errors accumulate so that the methods of classical geodesy are not adequate.

With the methods of physical geodesy (Stokes, 1849; Vening Meinesz, 1928) we try to find the geoid as it actually is. To do this we theoretically need measurements of gravity over the entire earth. Even if these measurements do not cover the whole world, we can nevertheless obtain an accuracy to within a few seconds. We can then relate the reference directions over any large area, provided we have measurements of gravity.

But even if we have related the reference directions at the different points, another problem remains to be solved: the effect of refraction.

The different lines of the geometric figures that we establish on the earth's surface are defined by rays of light. Because of atmospheric refraction the light rays are bent, and this bending occurs almost completely on a vertical plane. Although we correct for the effect of refraction, the proper amount of correction is very uncertain, and the errors in vertical angles (for long lines) can attain several seconds. It is important to note, however, that such errors diminish as the light rays move away from the earth's surface.

[1] This paper is based on a dissertation, "Geodetic Applications of Observations to the Moon, Artificial Satellites, and Rockets," presented to the Institute of Geodesy, Photogrammetry, and Cartography, of Ohio State University.

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These difficulties have led to a duality in the methods of geodesy. Determinations of positions are made separately, for the planimetry and for the elevations. The planimetry is based on a more or less geometric procedure, triangulation, while the vertical control is based on a physical method, precise levelling.

The methods presented here depend on observations of orbiting objects such as artificial satellites. These methods have the advantage that they use a unique system of reference direction which is not affected by the earth's gravitational field (or the geoid), and that the errors from refraction are reduced to a minimum. The mathematical tools are fairly simple, consisting mainly of analytical geometry in three dimensions.

In principle, the method involves our performing a three-dimensional space triangulation, with the observed objects (satellites) and the observing stations on the earth as the vertices. On this space-triangulation net we compute the positions of the stations. Since the observed object will usually be at a high altitude, errors from refraction are reduced to a minimum.

An important feature of the methods presented here is that instead of measuring angles to obtain directions, we determine them directly, in relation to a reference system defined by the stars. The relative positions of the stars, corrected for proper motion, aberration, and parallax, will be the same for all observers, everywhere on the earth. Therefore if we connect a system of reference to the stars, we can then define the system by the equator and the ecliptic of a certain epoch.

For practical reasons, however, we may also use a system of reference that is not fixed with respect to the stars but is moving; i.e., we use a system defined by the instantaneous equator and ecliptic. This motion (precession and nutation) actually will not affect the accuracy of the definition of our reference system. The accuracy depends only on the consistency of the positions of the stars, and the accuracy of their proper motions.

Finally, this system will be connected to a reference system on the earth. Since the two systems will have a relative motion, we must have continuous astronomic observations from fixed observatories, to connect the two systems.

Although the methods presented here were devised specifically for use with the artificial satellites, the same methods can be applied, with some modifications, to such targets as the moon, rockets, and even to terrestrial objects. Because these methods have such general applications, the presentation that follows will commonly use the word "object" rather than "satellite."

The coordinate systems

The several systems of coordinates and their transformations present various problems.

Elliptical coordinate systems.—The system commonly used in geodesy is that of ellipsoidal coordinates. A point Q is defined by projecting it along the normal to a reference ellipsoid. The projection Q' on the ellipsoid is defined by the latitude ϕ and longitude λ. The distance QQ' defines the height H of the point Q above the ellipsoid.

Some question exists whether the curved vertical should be used instead of the normal. For points near the surface of the earth, however, the differences are extremely small. For a height of 10 km, for example, the differences are 4 cm in ϕ, 0 in λ, 10^{-4} cm in H.

In an ideal system of coordinates, the reference ellipsoid would be centered at the center of the mass of the earth and the 3-axis would coincide with the mean axis of rotation. This is called the terrestrial ellipsoid (Heiskanen and Vening Meinesz, 1958). Since the center of mass of the earth is not known, this ideal system of terrestrial coordinates unfortunately cannot be realized. It is therefore necessary to substitute either the astronomic or the geodetic coordinate system.

The astronomic coordinates ϕ_A and λ_A are defined as the angles which the vertical makes with the equator and the meridian of Greenwich.

These coordinates are related to the geoid and can be obtained with an accuracy of the order of 0″1 by astronomic observations. They differ by several seconds from the terrestrial coordinates on account of the irregularities of the gravitational field. Gravimetrically corrected for the deflection of the vertical, the astronomic coordinates can come much closer to the terrestrial.

Terrestrial rectangular coordinates.—The geodetic coordinates ϕ_G and λ_G are defined in the same way as the terrestrial coordinates ϕ and λ, but they refer to a particular computation ellipsoid which may not be centered at the center of gravity of the earth, and may not be oriented appropriately. These coordinates are computed from geodetic surveys on a reference or computation ellipsoid, which is oriented at the origin of the geodetic system (Bomford, 1952).

For points referred to the same geodetic system, the relative positions will be correct within the accuracy of the survey. They will not be correct if the points do not refer to the same system.

The elliptical coordinates (ϕ, λ, H) are not convenient for points far from the earth's surface. For such points, it is much more convenient to use a system of rectangular coordinates.

The particular rectangular coordinate system $\mathbf{X}$, presented here, has its origin at the center of gravity of the earth (or of the terrestrial ellipsoid) and is oriented so that the 3-axis is directed toward the mean north pole, as defined by the International Latitude Service, and the 1–3 plane is parallel to the mean meridian of Greenwich (the meridian instrument at Greenwich does not lie in the 1–3 plane), as defined by the Bureau International de l'Heure (Stoyko, 1955).

This coordinate system is fixed with respect to the earth's surface, and the coordinates of any point on the earth are fixed and do not change with time, if we assume no movements of the crust.

A point is defined in this system by a vector X^i (X^1, X^2, X^3). This system is related to the ideal system of the geographic coordinates by the following formulas of transformation:

$$\begin{aligned} X^1&=(N+H)\cos\phi\cos\lambda=\rho\cos\beta\cos\lambda \\ X^2&=(N+H)\cos\phi\sin\lambda=\rho\cos\beta\sin\lambda \\ X^3&=[(1-e^2)N+H]\sin\phi=\rho\sin\beta \end{aligned} \qquad (1$$

where N=radius of curvature in prime vertical, e=eccentricity of the ellipsoid, ρ=radius vector, and β=geocentric latitude.

The first set of equations is more convenient, since tables (Perrier and Hasse, 1935; Army Map Service, 1944) may be obtained for the values of N.

Sidereal rectangular coordinates.—For points that do not rotate with the earth, it is convenient to have a coordinate system that does not rotate with the earth. This is the sidereal system. We can reach this, however, only by means of an intermediate system, the instantaneous terrestrial coordinate system $\mathbf{Y}$.

The axis of rotation of the earth is not fixed with respect to its surface. The motion of the true pole is studied by the International Latitude Service, which gives the coordinates of the instantaneous pole with respect to a mean pole (the same pole used to define the X^3-axis).

If we know the coordinates of the apparent (instantaneous) pole we can define another system $\mathbf{Y}$ of coordinates which uses this instantaneous axis as the Y^3-axis. The Y^3Y^1-plane contains the point where the mean meridian of Greenwich intersects the equator, which is also the X^1-axis, since the instantaneous zero meridian is thus defined by the Bureau International de l'Heure (B.I.H.) for the time UT1. (According to the decision of the International Astronomical Union at the 1955 meeting in Dublin, UT0 is the observed time; UT1 is the observed time corrected for the motion of the pole; and UT2 is UT1 corrected for the seasonal variations of the rotation of the earth.)

Let x and y (fig. 1) be the angular (spherical) coordinates of the instantaneous pole P with respect to the mean pole $\overline{P}$. We have taken X^1 as the primary axis, but since x and y are less than 1 second of arc, the result is practically the same.

The transformation from the $\mathbf{X}$ to the $\mathbf{Y}$ system is given by the expression

$$Y^i=M^i_{(XY)j}X^j$$

in which $M^i_{(XY)j}$ is the matrix:

$$M^i_{(XY)j}=\begin{pmatrix}\cos X^1Y^1 & \cos X^2Y^1 & \cos X^3Y^1 \\ \cos X^1Y^2 & \cos X^2Y^2 & \cos X^3Y^2 \\ \cos X^1Y^3 & \cos X^2Y^3 & \cos X^3Y^3\end{pmatrix}.$$

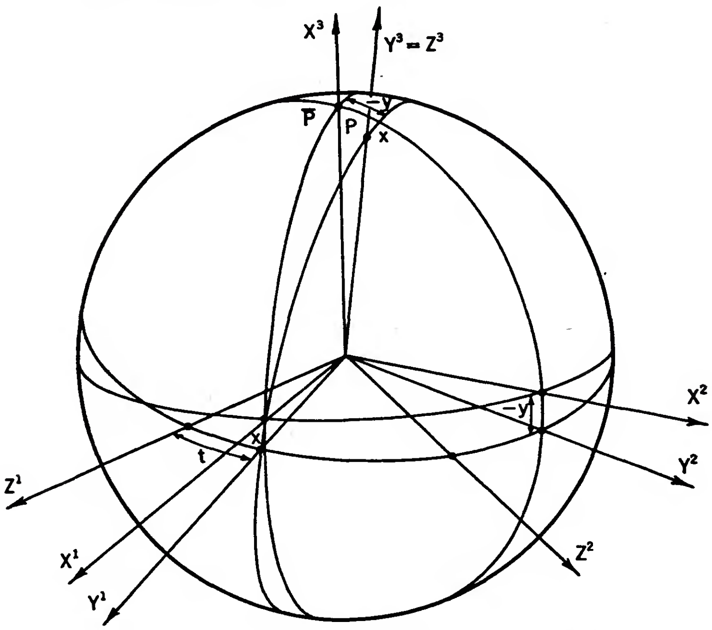

FIGURE 1.—Terrestrial and sidereal coordinate systems.

From the spherical triangles in figure 1 we obtain the formulas given in equation (2) and, consequently, the relation in equation (2a).

In the sidereal coordinate system **Z**, the Z^3-axis will coincide with the Y^3-axis, and the Z^1-axis will be directed toward the apparent vernal equinox ♈. The transformation from the **Y** to the **Z** system will be

$$Z^i = M^i_{(YZ)j} Y^j$$

with

$$M^i_{(YZ)j} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

t being the angle Y^1Z^1 or

$$\begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}. \qquad (3)$$

$$\begin{array}{lll} \cos X^1Y^1 = \cos x & \cos X^2Y^1 = \sin x \sin y & \cos X^3Y^1 = -\sin x \cos y \\ \cos X^1Y^2 = 0 & \cos X^2Y^2 = \cos y & \cos X^3Y^2 = +\sin y \\ \cos X^1Y^3 = \sin x & \cos X^2Y^3 = -\cos x \sin y & \cos X^3Y^3 = \cos x \cos y. \end{array} \qquad (2)$$

$$\begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} = \begin{pmatrix} \cos x & +\sin x \sin y & -\sin x \cos y \\ 0 & \cos y & +\sin y \\ \sin x & -\sin y \cos x & +\cos x \cos y \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}. \qquad (2a)$$

From the definition, the angle t is clearly the same as the Greenwich apparent sidereal time as defined by the B.I.H. (Stoyko, 1955).

We can go directly from the X- to the Z-coordinates, since

$$Z^i = M^i_{(YZ)_k} M^k_{(XY)_j} X^j \qquad (4)$$

or as in equation (4a).

The sidereal time t must be considered as an angle in space and not as elapsed time.

To find t we must use UT1 rather than UT2 since UT2 is corrected for seasonal variation in the speed of rotation of the earth and thus is not a true measure of the angle between Y^1 and Z^1. This correction (periodic) has an amplitude of about 0.03 or 0″.5.

We have now obtained a system of coordinates (**Z**) that is not affected by the rotation of the earth. This system is not fixed in space, however, since the axis of rotation of the earth, which defines our system, is moving under the influence of precession and nutation. There are two methods that could be applied here:

1. Keep the **Z** system and reduce all systems to the apparent positions. This reduction can be made easily by using the classical methods of reduction to apparent positions (Besselian Star Numbers or Independent Star Numbers). The necessary tables can be found in the various Ephemerides.
2. Use another system of coordinates, **W**, that will be defined as being the mean sidereal system at a certain fixed epoch T_0. This system will be defined as follows: The W^1 axis will be directed toward the mean equinox $\overline{\Upsilon}$ of T_0 and the W^3 axis will be directed toward the mean pole $\overline{P}$ of T_0.

The relation between the **Z** and the **W** systems will be obtained in two steps through a system which will be defined as being the **W** system but at epoch T_0; i.e., the epoch in which the **Z** system is defined. We will call this system $\tilde{\mathbf{Z}}$. This step is necessary since the motion of the polar axis is given in two parts, as precession and as nutation.

Let κ and ω be the angle between the line of intersection of the mean equator at T_0 and the mean equator at T with the W and $\tilde{Z}^2$ axes respectively (fig. 2). Let ν be the angle between the planes of the two mean equators (same figure). The transformation from the **W** to $\tilde{\mathbf{Z}}$ system will be given by the formula:

$$\tilde{Z}^i = M^i_{(W\tilde{Z})_j} W^j,$$

or by the rigorous expression of equation (5).

The values of κ, ω and ν are given as (Danjon, 1952):

$$\kappa = [23042''.53 + 139''.73(T_0 - 1.900) + 0''.06(T_0 - 1.900)^2]\,(T - T_0) + [30''.23 - 0''.27(T_0 - 1.900)]\,(T - T_0)^2 + 18''.00(T - T_0)^3$$

$$\omega = [23042''.53 + 139''.73(T_0 - 1.900) + 0''.06(T_0 - 1.900)^2]\,(T - T_0) + [109''.50 + 0''.39(T_0 - 1.900)]\,(T - T_0)^2 + 18''.32(T - T_0)^3$$

$$\nu = [20046''.85 - 85''.33(T_0 - 1.900) - 0''.37(T - 1.900)^2]\,(T - T_0) + [-42''.67 - 0''.37(T - 1.900)]\,(T - T_0)^2 - 41''.80(T - T_0)^3$$

where T_0 and T are expressed in units of a thousand tropical years.

$$\begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} \cos t \cos x & \cos t \sin x \sin y - \sin t \cos y & -\cos t \sin x \cos y - \sin t \sin y \\ \sin t \cos x & \sin t \sin x \sin y + \cos t \cos y & -\sin t \sin x \cos y + \cos t \sin y \\ \sin x & -\cos x \sin y & \cos x \cos y \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}. \qquad (4a)$$

$$\begin{pmatrix} \tilde{Z}^1 \\ \tilde{Z}^2 \\ \tilde{Z}^3 \end{pmatrix} = \begin{pmatrix} -\sin\kappa \sin\omega + \cos\kappa \cos\omega \cos\nu & -\cos\kappa \sin\omega - \sin\kappa \cos\omega \cos\nu & -\cos\omega \sin\nu \\ \sin\kappa \cos\omega + \cos\kappa \sin\omega \cos\nu & \cos\kappa \cos\omega - \sin\kappa \sin\omega \cos\nu & -\sin\omega \sin\nu \\ \cos\kappa \sin\nu & -\sin\kappa \sin\nu & \cos\nu \end{pmatrix} \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix}. \qquad (5)$$

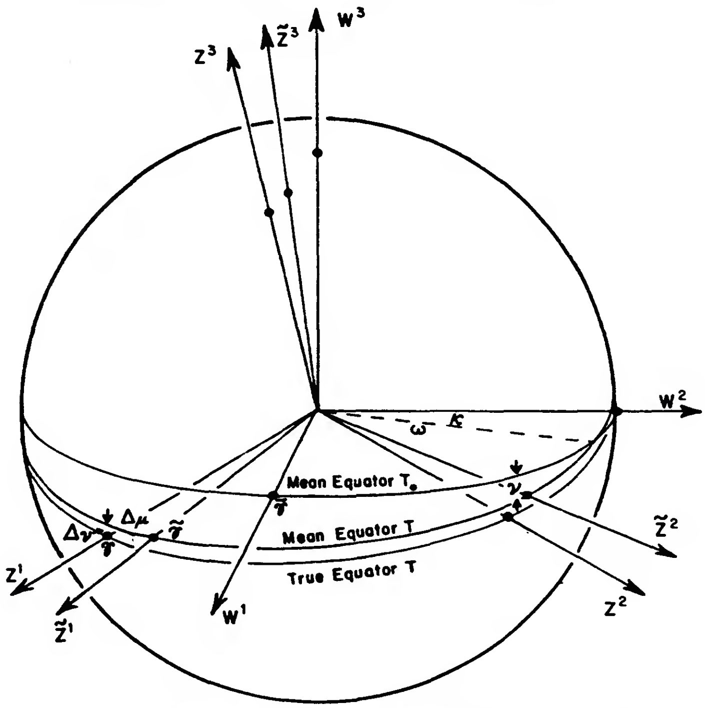

FIGURE 2.—Instantaneous and mean sidereal coordinate systems.

If we denote by $\Delta\mu$, $\Delta\nu$, $\Delta\epsilon$, respectively, the nutation in right ascension, declination, and obliquity, we get

$$Z^k = M^k_{(\tilde{Z}Z)_i} \tilde{Z}^i;$$

or with sufficient approximation, omitting terms of the order 10^{-9},

$$\begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 1 & -\Delta\mu & -\Delta\nu \\ \Delta\mu & 1 & -\Delta\epsilon \\ \Delta\nu & \Delta\epsilon & 1 \end{pmatrix} \begin{pmatrix} \tilde{Z}^1 \\ \tilde{Z}^2 \\ \tilde{Z}^3 \end{pmatrix}. \quad (5a)$$

The values of $\Delta\mu$, $\Delta\nu$, $\Delta\epsilon$, can be obtained from an almanac for the epoch T. Combining the two rotations we get

$$Z^k = M^k_{(WZ)_j} W^j$$

and

$$W^j = M^j_{(ZW)_k} Z^k,$$

where

$$M_{(ZW)} = M'_{(WZ)}.$$

We have now achieved a system of rectangular coordinates (**W**) centered at the center of gravity of the earth. This system is not fixed in space, but—what is more important—it has no rotation. We have also established the formulas for making the transformations from one system to another. All these systems are geocentric, which means that we should know the position of the center of gravity of the earth.

Geodetic rectangular coordinates.—Just as we were obliged to use a system of geodetic coordinates as substitutes for the geographic coordinates, so we are required to use a system of "geodetic rectangular coordinates" as a substitute for the terrestrial geocentric coordinates.

The geodetic rectangular coordinate system **x** is defined by the geodetic coordinates φ_G, λ_G, H_G given to our triangulation points and by the parameters of the computation ellipsoid used. If we compute the rectangular coordinates X^i from equation (1) by using φ_G, λ_G, H_G, we obtain exactly our geodetic rectangular coordinates. For each different geodetic system, we obtain a different system of rectangular coordinates.

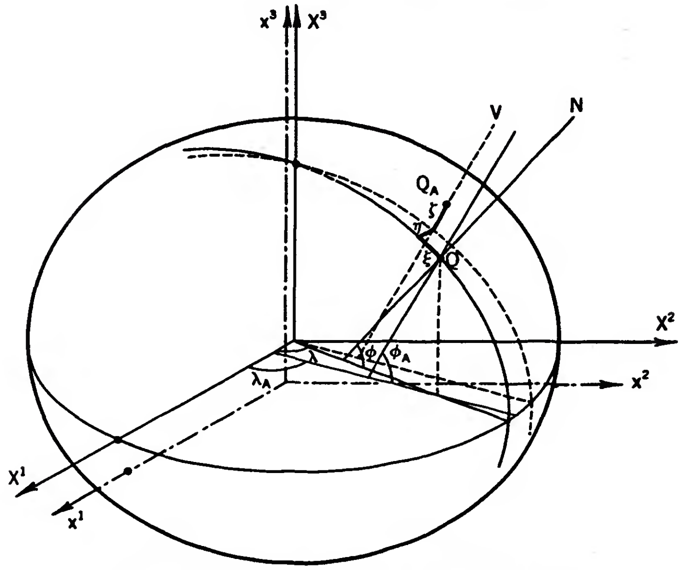

FIGURE 3.—Geodetic and terrestrial coordinate systems.

Let φ_A, λ_A, H_A be the coordinates of the origin of a geodetic system obtained by astronomic methods and leveling, and let ξ, η, ζ be the absolute deflections, so that

$$\xi=\varphi_A-\varphi$$

$$\eta=(\lambda_A-\lambda)\cos\varphi$$

$$\zeta=H_A-H.$$

In general, to the origin Q_0 of the system we assign the astronomic coordinates φ_A, λ_A, H_A. But to those coordinates corresponds another point Q_A on the terrestrial (geocentric) ellipsoid. Since the point Q is the same physical point on the earth, we shall have a displacement of the system of coordinates. This displacement will be a parallel translation since at the origin of the system we make the theoretical normal at Q_A correspond with the observed vertical at Q_0. These two lines are parallel (fig. 3), while the direction of the X^3 axis is identical with the direction of the pole (which is observed directly). We assume here that there are no errors in the determination of the azimuth.

If we express ξ and η in linear units by using the radii $\rho+H$ and $N+H$, the total translation will be $(\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$. In the **X** system, the coordinates of the new origin will be X_C^i and the transformation is given by matrices (6) and (7).

It should be noted that the same formulas could be obtained by differentiating equation (1), and that equations (6) and (7) apply only for small values of ξ, η, ζ.

$$\begin{bmatrix} X_C^1 \\ X_C^2 \\ X_C^3 \end{bmatrix}=\begin{bmatrix} \sin\varphi_0\cos\lambda_0 & \sin\lambda_0 & -\cos\varphi_0\cos\lambda_0 \\ \sin\varphi_0\sin\lambda_0 & -\cos\lambda_0 & -\cos\varphi_0\sin\lambda_0 \\ -\cos\varphi_0 & 0 & -\sin\varphi_0 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{6}$$

and

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}=\begin{bmatrix} \sin\varphi_0\cos\lambda_0 & \sin\varphi_0\sin\lambda_0 & -\cos\varphi_0 \\ \sin\lambda_0 & -\cos\lambda_0 & 0 \\ -\cos\varphi_0\cos\lambda_0 & -\cos\varphi_0\sin\lambda_0 & -\sin\varphi_0 \end{bmatrix}\begin{bmatrix} X_C^1 \\ X_C^2 \\ X_C^3 \end{bmatrix}. \tag{7}$$

The parallelism between the geocentric terrestrial system (terrestrial ellipsoid) and the geodetic system (computation ellipsoid) is correct not only if we use the astronomic values of φ, λ, and azimuth A, but also if we apply corrections to the astronomic coordinates for deflection of the vertical, provided the astronomic azimuth is *also* corrected for the Laplace equation. In this case ξ, η, ζ are the residuals from the absolute deflections.

So if x^i are the coordinates in the geodetic system we have the equation,

$$x^i = X^i - X^i_C. \tag{8}$$

Until now we have assumed that the astronomic coordinates at the origin of the system were correct. An error in φ_A, λ_A or H_A will have no effect on the parallelism between the two systems (at least to the first order), since it will introduce only a different displacement. However, an error in the azimuth, which is related to the longitude, does affect this parallelism.

Obviously an error dA in the azimuth at the initial point of the system will result in a rotation of the system of coordinates around the vertical, by the amount dA (in this case dA is the error in the azimuth plus the effect of the error in λ). We will compute the effect of this rotation.

First we notice that since the axis around which the system rotates does not pass from the origin, we will have also a translation. The rotation matrix can be obtained as a product of five rotations, expressed by the matrix (8a).

Since the angle dA will be very small (of the order of $1''$), we find the matrix M_A for the rotation as follows:

We know that a vector r^i after an infinitesimal rotation dA^i will become $r^i_A = r^i + dA^i \times r^i$.

If we consider a matrix M whose columns are the vectors $r^i_1, r^i_2 \ldots r^i_j$, the matrix M_A of the same vectors after the rotation will be

$$M_A = M + dA^i \times M. \tag{9}$$

The cross product is simply the matrix whose columns are the vectors of the cross product $dA^i \times r^i_j$.

If M has as columns the three unit vectors of the $X^i - X^i_C$ system (the **X** system after the

$$M_A = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\varphi & 0 & \cos\varphi \\ 0 & 1 & 0 \\ -\cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} \cos dA & \sin dA & 0 \\ -\sin dA & \cos dA & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\varphi & 0 & -\cos\varphi \\ 0 & 1 & 0 \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8a}$$

parallel displacement), it is clear that M_A will be the matrix of rotation so that

$$(X^i - X_C^i) - x_0^i = M^i_{(A)j}(x^j - x_0^j), \qquad (10)$$

x_0^i being the coordinates of the initial point of the geodetic system.

But then $M=I$ (the unit matrix) so that equation (9) gives the matrix

$$M^i_{(A)j} = \begin{pmatrix} 1 & -dA^3 & dA^2 \\ dA^3 & 1 & -dA^1 \\ -dA^2 & dA^1 & 1 \end{pmatrix}. \qquad (11)$$

In our case if dA is the magnitude of the error in A, then

$$dA^i = dA(-\cos\phi_0 \cos\lambda_0, -\cos\phi_0 \sin\lambda_0, -\sin\phi_0)$$

and we obtain the matrix in equation (12).

This matrix is not orthonormal since the second and higher order terms of dA have been disregarded. However, since dA is expected to be less than 1″, which is about 5×10^{-6}, the discrepancies will arise in the 11th decimal.

If we then introduce the values of $M^i_{(A)j}$ to equation (10), we get:

$$\begin{aligned} X^1 &= X_C^1 + x^1 + [\sin\varphi_0(x^2 - x_0^2) - \cos\varphi_0 \sin\lambda_0(x^3 - x_0^3)]\, dA, \\ X^2 &= X_C^2 + x^2 + [-\sin\varphi_0(x^1 - x_0^1) + \cos\varphi_0 \cos\lambda_0(x^3 - x_0^3)]\, dA, \\ X^3 &= X_C^3 + x^3 + [\cos\varphi_0 \sin\lambda_0(x^1 - x_0^1) - \cos\varphi_0 \cos\lambda_0(x^2 - x_0^2)]\, dA. \end{aligned} \qquad (13)$$

The formulas are correct to at least 10^{-10}.

The computation ellipsoid (and thus the geodetic system), as defined by the origin of the system, will be parallel to the terrestrial ellipsoid and the terrestrial system, if we assume no error in the astronomic azimuth. Since the triangulation is calculated on the same computation ellipsoid, there is apparently no reason for any tilt between the computation and the terrestrial ellipsoid.

Indeed, that would be the case if the coordinates of the different points were computed with a consistent geometric method. However, the measurements are made with reference to the geoid, while the computations are made on the ellipsoid. Furthermore, the heights are measured by leveling and are measured from the geoid. This means that there is a kind of distortion in the triangulation net.

If the computation ellipsoid is very near the geoid over the area of the triangulation, the distortion will be minimum. It is possible, however, that the computation ellipsoid may be tilted with respect to the mean geoid over the area. This would happen if, on account of local attraction at the origin of the system, the geoid were very rough and thus the true vertical were tilted from a mean vertical by an amount $d\xi$ (and $d\eta$) (fig. 4).

Since heights of points are measured from the geoid, we must tilt the computation ellipsoid so that the computed position Q_1' is on the true point Q_1. This means a tilt around point Q_0 by $d\xi$ (and $d\eta$). This tilt will bring an inconsistency in that the computed direction of the pole (axis x^3) will not be the one that has been observed.

We must notice, however, that the angle $d\xi$ (and $d\eta$) will be zero or very near zero, if we have corrected the heights to those corresponding to the computation ellipsoid, e.g., by making astronomic leveling; or if we have corrected for the deflection of the vertical and hence for the local attraction by, e.g., a gravity survey around the initial point of the system.

Let $d\xi$ and $d\eta$ be the magnitude of the tilt. The vectors of rotation $d\xi^i$ $d\eta^i$ will be

$$d\xi^i = d\xi(\sin\lambda_0, -\cos\lambda_0, 0),$$

$$d\eta^i = d\eta(-\sin\varphi_0, \cos\lambda_0, -\sin\varphi_0, \sin\lambda_0, \cos\varphi_0),$$

with positive direction to the west for $d\xi^i$ and to the north for $d\eta^i$.

$$M^i_{(A)j} = \begin{pmatrix} 1 & +dA \sin\varphi_0 & -dA\cos\varphi_0 \sin\lambda_0 \\ -dA\sin\varphi_0 & 1 & +dA\cos\varphi_0\cos\lambda_0 \\ dA\cos\varphi_0\sin\lambda_0 & -dA\cos\varphi_0\cos\lambda_0 & 1 \end{pmatrix}. \qquad (12)$$

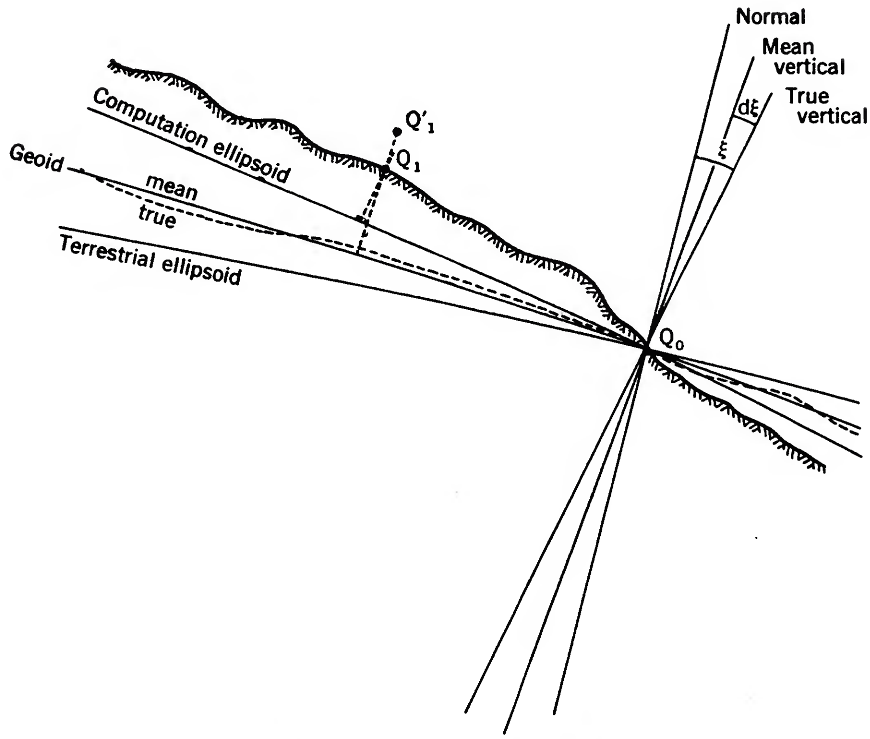

FIGURE 4.—Tilt of the computation ellipsoid.

We can compute the rotation matrices $M_{(\xi)}$ and $M_{(\eta)}$ with the help of equation (11), replacing dA^i with $d\xi^i$ and $d\eta^i$.

We then obtain equations (14) and (15).

Thus if we consider only the effect of $d\xi$ or $d\eta$ we will have

$$X^i - X^i_c - x^i_0 = M^i_{(\xi)j}(x^j - x^j_0)$$

$$X^i - X^i_c - x^i_0 = M^i_{(\eta)j}(x^j - x^j_0).$$

The total rotation from the effect of dA, $d\xi$, and $d\eta$, will be

$$M_{(A,\xi,\eta)} = M_{(A)} \cdot M_{(\xi)} \cdot M_{(\eta)}$$

$$M^i_{(\xi)j} = \begin{bmatrix} 1 & 0 & -d\xi\cos\lambda_0 \\ 0 & 1 & -d\xi\sin\lambda_0 \\ d\xi\cos\lambda_0 & d\xi\sin\lambda_0 & 1 \end{bmatrix}, \tag{14}$$

$$M^i_{(\eta)j} = \begin{bmatrix} 1 & -d\eta\cos\varphi_0 & -d\eta\sin\varphi_0\sin\lambda_0 \\ d\eta\cos\varphi_0 & 1 & d\eta\sin\varphi_0\cos\lambda_0 \\ d\eta\sin\varphi_0\sin\lambda_0 & -d\eta\sin\varphi_0\cos\lambda_0 & 1 \end{bmatrix}. \tag{15}$$

(since we have eliminated second order terms the order of multiplication does not affect the result), or as in equation (16).

If we include all three errors of orientation, dA, $d\xi$, $d\eta$, then equation (10) will be written as

$$X^i - X^i_C - x^i_0 = M^i_{(A\xi\eta)j}(x^j - x^j_0) \qquad (17)$$

or

$$X^i = x^i + X^i_C + (M_{A\xi\eta} - I)(x^j - x^j_0). \qquad (17a)$$

Grouping the terms, if g^j is the vector (dA, $d\xi$, $d\eta$) we get

$$X^i = x^i + X^i_C + G^i_j g^j \qquad (18)$$

or

$$\begin{aligned} X^1 = x^1 + X^1_C + [&\sin\varphi_0(x^2 - x^2_0) - \\ &\cos\varphi_0 \sin\lambda_0(x^3 - x^3_0)]dA - \\ &[\cos\lambda_0(x^3 - x^3_0)]\, d\xi - [\cos\varphi_0(x^2 - x^2_0) + \\ &\sin\varphi_0 \sin\lambda_0(x^3 - x^3_0)]\, d\eta, \end{aligned} \qquad (18a)$$

$$\begin{aligned} X^2 = x^2 + X^2_C + [&-\sin\varphi_0(x^1 - x^1_0) + \\ &\cos\varphi_0 \cos\lambda_0(x^3 - x^3_0)]\, dA - \\ &[\sin\lambda_0(x^3 - x^3_0)]\, d\xi + [\cos\varphi_0(x^1 - x^1_0) + \\ &\sin\varphi_0 \cos\lambda_0(x^3 - x^3_0)]\, d\eta, \end{aligned} \qquad (18b)$$

$$\begin{aligned} X^3 = x^3 + X^3_C + [&\cos\varphi_0 \sin\lambda_0(x^1 - x^1_0) - \\ &\cos\varphi_0 \cos\lambda_0(x^2 - x^2_0)]\, dA + \\ &[\cos\lambda_0(x^1 - x^1_0) + \sin\lambda_0(x^2 - x^2_0)]\, d\xi + \\ &[\sin\varphi_0 \sin\lambda_0(x^1 - x^1_0) - \\ &\sin\varphi_0 \cos\lambda_0(x^2 - x^2_0)]\, d\eta. \end{aligned} \qquad (18c)$$

Let us now consider the effect of an error in scale. Since triangulation schemes are scaled by geodetic base lines, nonconnected triangulations may be at different scales.

Then, again eliminating second-order terms, we should add $\epsilon\,(x^i - x^i_0)$ to equation (18). With Δx^j used for $(x^j - x^j_0)$, the final equations of transformation will become

$$X^i = x^i + X^i_C + G^i_j g^j + \epsilon \Delta x^i, \qquad (19)$$

or

$$\begin{aligned} X^1 = x^1 + X^1_C + (&\sin\varphi_0 \Delta x^2 - \\ &\cos\varphi_0 \sin\lambda_0 \Delta x^3)dA - (\cos\lambda_0 \Delta x^3)d\xi - \\ &(\cos\varphi_0 \Delta x^2 + \sin\varphi_0 \sin\lambda_0 \Delta x^3)d\eta + \epsilon\Delta x^1, \end{aligned} \qquad (19a)$$

$$\begin{aligned} X^2 = x^2 + X^2_C + (&-\sin\varphi_0 \Delta x^1 + \\ &\cos\varphi_0 \cos\lambda_0 \Delta x^3)dA - (\sin\lambda_0 \Delta x^3)d\xi + \\ &(\cos\varphi_0 \Delta x^1 + \sin\varphi_0 \cos\lambda_0 \Delta x^3)d\eta + \epsilon\Delta x^2, \end{aligned} \qquad (19b)$$

$$\begin{aligned} X^3 = x^3 + X^3_C + (&\cos\varphi_0 \sin\lambda_0 \Delta x^1 - \\ &\cos\varphi_0 \cos\lambda_0 \Delta x^2)dA + (\cos\lambda_0 \Delta x^1 + \\ &\sin\lambda_0 \Delta x^2)d\xi + (\sin\varphi_0 \sin\lambda_0 \Delta x^1 - \\ &\sin\varphi_0 \cos\lambda_0 \Delta x^2)d\eta + \epsilon\Delta x^3; \end{aligned} \qquad (19c)$$

or also,

$$X^i = x^i + X^i_C + (M_{A\xi\eta} - I)\Delta x^j + \epsilon \cdot \Delta x^i. \qquad (20)$$

Finally, we have another group of systems of rectangular coordinates, which are parallel to the geocentric but have their origin at a particular station Q. Such a system will be called "topocentric" and will be designated by a prime accent. If we refer to terrestrial coordinates, the transformation is given by the relation

$$X'^i = X^i - X^i_Q.$$

Similarly, in geodetic coordinates we have

$$x'^i = x^i - x^i_Q.$$

$$M^i_{(A\xi\eta)j} = \begin{Bmatrix} 1 & dA\sin\varphi_0 - d\eta\cos\varphi_0 & -dA\cos\varphi_0\sin\lambda_0 - d\xi\cos\lambda_0 - d\eta\sin\varphi_0\sin\lambda_0 \\ -dA\sin\varphi_0 + d\eta\cos\varphi & 1 & dA\cos\varphi_0\sin\lambda_0 - d\xi\sin\lambda_0 + d\eta\sin\varphi_0\cos\lambda_0 \\ dA\cos\varphi_0\sin\lambda_0 + d\xi\cos\lambda_0 + d\eta\sin\varphi_0\sin\lambda_0 & -dA\cos\varphi_0\sin\lambda_0 + d\xi\sin\lambda_0 - d\eta\sin\varphi_0\cos\lambda_0 & 1 \end{Bmatrix}. \qquad (16)$$

Accuracy of the transformations.—The transformation from one coordinate system to another requires that we use a number of parameters that are derived from observations and theory. We shall try to estimate the expected accuracy of the transformed coordinates, using the accuracy with which the parameters are known.

In making transformations from the **X** system to the **Y**, from the expression

$$Y^i = M^i_{(XY)_j} X^j$$

we get the equation

$$dY^i = dM^i_{(XY)_j} X^j. \tag{21}$$

Taking the differentials of the elements of the matrix $M^i_{(XY)_j}$, we obtain equation (21a). Since the values of x and y are small, we can eliminate the second-order terms so that

$$dM^i_{(XY)_j} = \begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & dy \\ dx & -dy & 0 \end{pmatrix},$$

and therefore

$$\begin{pmatrix} dY^1 \\ dY^2 \\ dY^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & dy \\ dx & -dy & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}. \tag{22}$$

It is not easy to estimate the accuracy with which the coordinates x and y of the pole are given. We accept $\pm 0\overset{''}{.}02$ as a measure of the accuracy (Melchior, 1954, p. 36). With this value for dx and dy we have, for points on the surface of the earth, errors (expected) of about 0.40 meters for the transformation to the **Y** system.

When we perform the transformation from the **Y** (or **X**) to the **Z** system, we have

$$dZ^i = dM^i_{(YZ)_j} Y^j. \tag{23}$$

We evaluate:

$$dM^i_{(XY)_j} = \begin{pmatrix} -\sin t dt & -\cos t & 0 \\ \cos t dt & -\sin t dt & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore

$$\begin{pmatrix} dZ^1 \\ dZ^2 \\ dZ^3 \end{pmatrix} = \begin{pmatrix} -\sin t dt & -\cos t dt & 0 \\ \cos t dt & -\sin t dt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}. \tag{24}$$

This formula is correct if Y^i has no errors. But from equation (21) we see that Y^i has the error dY^i so that equation (23) must be written as

$$dZ^i = dM^i_{(YZ)_j} Y^j + M^i_{(YZ)_j} dY^j,$$

or, if we replace dY^i from equation (21), as

$$dZ^i = dM^i_{(YZ)_j} Y^j + M^i_{(YZ)_k} dM^k_{(XY)_j} X^j.$$

Since $Y^i \simeq X^i$,

$$dZ^i = (dM^i_{(YZ)_j} + M^i_{(YZ)_k} dM^k_{(XY)_j}) X^j = dM^i_{(XZ)_j} X^j.$$

Or we have the matrix shown in (24a).

$$dM^i_{(XY)_j} = \begin{pmatrix} -\sin x dx & \sin y \cos x dx + \cos y \sin x dx & \sin x \cos y dy - \cos y \cos x dx \\ 0 & -\sin y dy & \cos y dy \\ \cos x dx & -\cos x \cos y dy + \sin x \sin y dx & -\cos x \sin y dy - \sin x \cos y dx \end{pmatrix}. \tag{21a}$$

$$M^i_{(YZ)_k} dM^k_{(XY)_j} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & dy \\ dx & -dy & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\cos t dx - \sin t dy \\ 0 & 0 & -\sin t dx + \cos t dy \\ dx & -dy & 0 \end{pmatrix}. \tag{24a}$$

Therefore

$$dM^i_{(XZ)_j} = \begin{pmatrix} -\sin t dt & -\cos t dt & -\cos t dx - \sin t dy \\ \cos t dt & -\sin t dt & -\sin t dx + \cos t dy \\ dx & -dy & 0 \end{pmatrix}.$$

Or, finally we have the form shown in equation (25).

Note that we could obtain the same result by taking the differentials of the elements of the matrix $M\{_{XZ}\}$ from equation (4) and eliminating the terms of second order.

The study of the accuracy of the angle t is very complicated. The apparent sidereal time of Greenwich is the hour angle of the apparent equinox ♈. It is related to the rotation of the earth and can be determined directly by astronomic observations at Greenwich.

In practice, however, we use Universal Time from which we find the apparent sidereal time at Greenwich by using the special tables in the ephemerides. This transformation is based on Newcomb's tables. Universal Time itself is obtained in the opposite manner, i.e., from the apparent sidereal time as determined from astronomic observations which have been transformed to Universal Time (mean time) with the same tables.

At first glance we might think that since the apparent sidereal time is determined directly (the use of the mean time as an intermediate stage has no effect) it is given with the accuracy of the observations themselves. This is not true, however, mainly because the observed values of the sidereal time have been smoothed to make them agree more closely with the time given by the clocks, to obtain a time as nearly uniform as possible. This smoothing eliminates the effect of changes in the rotation of the earth, at least those of short period.

It is obvious that the effect of the secular deceleration of the earth, as well as of the effect of the term B of fluctuation (Jones, 1939), is zero, since both have an extremely slow effect and UT is not corrected, for the moment, for this effect. However, corrections for the seasonal variations are applied according to the resolutions of the International Astronomical Union at the Dublin assembly. These corrections (ΔTs) are published in advance in the Bulletin Horaire. Since we do not require a uniform time, but the true angle of the point ♈, we must not use UT2 and convert it to the sidereal time but, instead, we must use UT1 which is not corrected for ΔTs. Since the radio time signals refer to UT2, we must apply the correction $-\Delta Ts$ to obtain UT1.

In analyzing the errors, we find that they result from errors in the astronomic observations made at the observatories of the B.I.H., and from the very short period irregularities of the rotation of the earth. Since we smooth our results we do not know how much of the error derives from each of these two categories.

We assume that the B.I.H. can provide the sidereal time with an accuracy better than $\pm 0^s.007$. How accurately we can determine the time of an observation is another question.

Expressing the estimated errors in radians, we have

$$dx = dy = \pm 0.1 \times 10^{-6}$$

$$dt = \pm 0.5 \times 10^{-6}$$

so that equation (25) becomes equation (25a).

$$\begin{bmatrix} dZ^1 \\ dZ^2 \\ dZ^3 \end{bmatrix} = \begin{bmatrix} -\sin t\,dt & -\cos t\,dt & -\cos t\,dx - \sin t\,dy \\ \cos t\,dt & -\sin t\,dt & -\sin t\,dx + \cos t\,dy \\ dx & -dy & 0 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}. \tag{25}$$

$$\begin{bmatrix} dZ^1 \\ dZ^2 \\ dZ^3 \end{bmatrix} = 10^{-6} \begin{bmatrix} \mp 0.5 \sin t & \mp 0.5 \cos t & \mp 0.1(\cos t + \sin t) \\ \pm 0.5 \cos t & \mp 0.5 \sin t & \pm 0.1(\cos t - \sin t) \\ \pm 0.1 & \mp 0.1 & 0 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}. \tag{25a}$$

Or

$$\begin{bmatrix} dZ^1 \\ dZ^2 \\ dZ^3 \end{bmatrix} = \begin{bmatrix} -X^1 \sin t - X^2 \cos t & -X^3 \cos t & -X^3 \sin t \\ X^1 \cos t - X^2 \sin t & X^3 \sin t & X^3 \cos t \\ 0 & X^1 & X^2 \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \end{bmatrix}. \tag{26}$$

For a point at $\phi=40°$, $\lambda=45°$, $H=0$ and for a sidereal time $t=3^h$, we get a total uncertainty in position of about 2.6 meters.

Equation (25) may also be written in the form of equation (26), or

$$dZ^t=M_t\begin{pmatrix}dt\\ dx\\ dy\end{pmatrix}.$$

If $V\{t, x, y\}$ is the variance of t, x, y, then the variance $V\{Z^t\}$ will be

$$V\{Z^t\}=M_t\cdot V\{t,x,y\}\cdot M_t'. \qquad (27)$$

Disregarding the errors in x and y, we obtain equation (28) where σ_t is the standard error in t, of the order of 0.5×10^{-6}.

$$V\{Z^t\}=\sigma_t^2\begin{bmatrix}(-X^1\sin t-X^2\cos t)^2 & (-X^1\sin t-X^2\cos t)(X^1\cos t-X^2\sin t) & 0\\ (-X^1\sin t-X^2\cos t)(X^1\cos t-X^2\sin t) & (X^1\cos t-X^2\sin t)^2 & 0\\ 0 & 0 & 0\end{bmatrix}. \qquad (28)$$

In transforming from the **Z** to the **W** system or vice versa, one should expect to introduce errors arising from errors in the values of the astronomical constants (precession, etc.). This source of error is more apparent than real, however, since such errors will be eliminated when we determine from observations the position of the mean pole and the sidereal time.

Determination of directions.—Since the stars lie at so great a distance from the earth, the direction of a star is the same from the center of the earth as from any station on its surface.

The directions of the stars are given in the various catalogs or ephemerides in the **W** or **Z** system, as right ascension (R.A.) and declination. Therefore we know the direction of the line connecting the center of the earth with the star; this direction is the same as that of the line connecting any station on the earth with the star.

In determining directions, if we use the **Z** system of coordinates we must use the true positions of the stars for the time of observation, i.e., referred to the equinox of the epoch. This direction (defined with α and δ) is the true and not the apparent. To get the apparent direction we must correct for aberration as well, or use the apparent positions of the stars.

If we use the **W** system of coordinates, we must use the mean positions of the stars for the epoch T_0. The mean positions must be corrected only for the proper motions of the stars, and if we need the apparent directions we must correct also for aberration.

The fact that a star has the same (known) direction from all stations on the earth indicates a method by which we can determine the direction of an object from a station in the sidereal system.

Let Q be the station and S be the object (fig. 5). Also let the object be in front of a

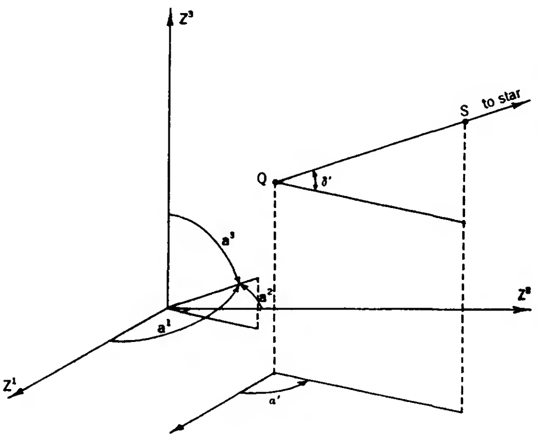

FIGURE 5.—Directions in the sidereal coordinate system.

fictitious star whose topocentric position is (α', δ'). It is obvious that the direction of the line QS will be given by α' and δ'.

If the direction cosines of QS in the $\mathbf{Z}$ system are

$$l^1 = \cos a^1$$

$$l^2 = \cos a^2$$

$$l^3 = \cos a^3,$$

we have the following simple relations between l^i and α', δ':

$$l^1 = \cos \delta' \cos \alpha'$$
$$l^2 = \cos \delta' \sin \alpha' \qquad (29)$$
$$l^3 = \sin \delta'$$

and

$$\sin \delta' = l^3$$
$$\tan \alpha' = \frac{l^2}{l^1}. \qquad (30)$$

Using equation (3) we can find the direction cosines m^i in the $\mathbf{Y}$ system by making Z^i a unit vector. We then obtain

$$m^1 = \cos \delta' \cos (\alpha' - t),$$
$$m^2 = \cos \delta' \sin (\alpha' - t),$$
$$m^3 = \sin \delta'.$$

When we introduce $\theta = \alpha - t = -(t - \alpha) = \text{GHA}$, we obtain

$$m^1 = \cos \delta' \cos \theta',$$
$$m^2 = \cos \delta' \sin \theta', \qquad (31)$$
$$m^3 = \sin \delta',$$

and

$$\tan \theta' = \frac{m^2}{m^1}, \qquad (32)$$
$$\sin \delta' = m^3.$$

Similarly, we get the direction cosines n^i in the $\mathbf{X}$ system by using equation (2). We obtain

$$n^1 = +\sin x \sin \delta' + \cos x \cos \delta' \cos \theta',$$
$$n^2 = -\cos x \sin y \sin \delta' + \sin x \sin y \cos \delta' \cos \theta' + \cos y \cos \delta' \sin \theta', \qquad (33)$$
$$n^3 = +\sin \theta' \sin y \cos \delta' - \sin x \cos y \cos \delta' \cos \theta' + \cos x \cos y \sin \delta',$$

and

$$\tan \theta' = \frac{n^2 \cos y + n^3 \sin y}{n^1 \cos x + \sin x \, n^2 \sin y - \sin x \, n^3 \cos y},$$
$$\sin \delta' = n^1 \sin x - \cos x \, n^2 \sin y + \cos x \, n^3 \cos y. \qquad (34)$$

To obtain directions in the $\mathbf{X}$ system, we get the direction cosines q^i in the geodetic system $\mathbf{X}$ by using equation (17) in the form

$$n^i = M^i_{(A\xi\eta)j} q^j, \qquad (35)$$

or

$$q^1 = n^1 + (-n^2 \sin \varphi_0 + n^3 \cos \varphi_0 \sin \lambda_0) dA + (n^3 \cos \lambda_0) d\xi + (n^2 \cos \varphi_0 + n^3 \sin \varphi_0 \sin \lambda_0) d\eta, \qquad (36a)$$

$$q^2 = n^2 + (n^1 \sin \varphi_0 - n^3 \cos \varphi_0 \sin \lambda_0) dA + (n^3 \sin \lambda_0) d\xi + (-n^1 \cos \varphi_0 - n^3 \sin \varphi_0 \cos \lambda_0) d\eta, \qquad (36b)$$

$$q^3 = n^3 + (-n^1 \cos\varphi_0 \sin\lambda_0 + n^2 \cos\varphi_0 \sin\lambda_0)dA +$$
$$(-n^1 \cos\lambda_0 - n^2 \sin\lambda_0)d\xi +$$
$$(n^1 \sin\varphi_0 \sin\lambda_0 - n^2 \sin\varphi_0 \sin\lambda_0)d\eta. \qquad (36c)$$

The big advantage of using the sidereal system of coordinates lies in the fact that we can get the direction of an object in this system directly from a photographic method suggested by Väisälä (1946).

The object is photographed with a metric camera, with the stars as background. Using the methods of photographic astrometry we find the topocentric apparent R.A. α' and declination δ' of the object. These values of α' and δ' then define the direction of the line *Q-S*. By using formulas (29) we can find the direction of the line between station and object with respect to the **Z** system of coordinates.

If we wish we can determine the direction in the **Y**, **X**, or **x** system by using formulas (31), (33), or (36), provided we know the parameters that relate these systems with the **Z** system, i.e., $t, x, y, dA, d\xi$, and $d\eta$. The effect of these parameters is small but we must know the time of the observation in order to determine t to better than 10 msec to have positional errors smaller than 5 meters. However, as another alternative we can fix the camera in the **X** (or the **x**) system, determine the orientation of the camera, using the stars in the same **X** system, and then determine the direction of the object, knowing the orientation of the camera. In this case the time of the observation is not needed.

A source of error is the possible error in the positions of the stars used as reference, as given in the various star catalogs.

Since the FK3 catalog (Dritter Fundamentalkatalog, 1937, 1938) is used for the definition of time, it must also be used for computing the apparent places of stars.

When we say the sidereal time is the hour angle of point ♈, we must make it clear that the point ♈ itself is not observed. The stars are observed; and knowing the R.A. of the stars we can find the hour angle of ♈. In other words, our **Z** system of coordinates is defined by the coordinates of a number of stars from one catalog and by the constants of precession, nutation, and aberration as well as by the proper motions that we have assumed. We might more correctly say that the point whose hour angle gives the sidereal time is not the intersection of the planes of the equator and the ecliptic, but the mean of the points defined as being $-\alpha_i$ from the stars.

So if we change the catalog from which we take our data, we change also our system of reference and our **Z** system. Corrections exist which enable one to shift from one catalog to another, and the same corrections should be applied to change from the **Z** system defined by the one catalog to the system defined by another.

Unfortunately the FK3 catalog that should be used contains only 1,500 stars, which are not always enough for our purpose. If we use a camera with long focus and a small field of view it is very probable that we will not observe a large enough number of stars included in FK3. For this reason we will be obliged to use additional stars that do not belong to FK3, and must reduce their positions to the FK3 system.

It is very difficult to say what the degree of accuracy is within the FK3 system. The internal errors are mainly accidental. Some systematic errors exist, resulting from errors in the astronomical constants, but these will be eliminated, since the **Z** system is defined with the same constants (the same errors with opposite sign are also made in the determination of the time). This is also true for the errors $\Delta\alpha_\alpha$ that are partly eliminated in the same way.

The observations

Although for orbiting objects the dynamic quantities (e.g., velocities) can be measured, this discussion will deal only with the measurable geometric quantities, i.e., distances and angles or directions.

Measurement of distances.—The only possible way to measure directly the distance to the object seems to be to use electromagnetic waves, or radar. The electromagnetic waves are reflected from the object and we measure the time needed for the waves to travel the double distance. Knowing the velocity of the electromagnetic waves, we can find the distance.

Better results can be obtained if the object carries a receiver-transmitter (transponder) to send back the signal. Both the pulse system (Shoran, Hiran) (Canada, 1955) and the phase system (Raydist) (Comstock and Hastings, 1952) can be used.

The accuracy of the distances thus measured depends on two things: the accuracy with which we can measure the small time interval the wave needs to travel; and the accuracy with which we know the velocity of the wave in the medium traveled. The distance r will be given by the equation

$$r=\frac{1}{2}V_m\tau \tag{37}$$

where V_m is the mean velocity along the path and τ the time interval.

Differentiating, we obtain:

$$dr=\frac{1}{2}V_m d\tau+\frac{1}{2}\tau dV_m,$$

or

$$dr=\frac{1}{2}V_m d\tau+\frac{dV_m}{V_m}r. \tag{38}$$

The value of V_m is about 0.3×10^9 m/sec, and if τ can be measured within ±0.02 msec, the distances will be given within ±3 meters.

An error dV_m causes an error proportional to the distance. V_m is computed from the value of c (velocity in vacuum) by applying a correction for the index of refraction. The latest value of c, accepted by the International Association of Geodesy at Toronto in 1957, gives $c=299{,}792.5\pm0.4$ km/sec, which gives a value of $dc/c=1.3\times10^{-6}$. However, the value of dV_m/V_m will be higher than that and probably of the order of 10^{-5}, because of uncertainties in the index of refraction. Furthermore, the curvature of the path will cause additional errors.

Finally, we must consider possible errors in the timing of the observations if the object is moving (compare with p. 112, where this problem is treated for the measurement of directions). Since time is a parameter, we will include the effect of an error in timing in the errors of the observed quantities, by using time as the independent variable.

If $\dot{r}$ is the change of the observed distance per unit of time and σ_T^2 the variance of timing, the variance of r will be

$$\sigma_{r_0}^2=\dot{r}^2\sigma_T^2,$$

and this value should be added to the variance σ_r^2 as given from an analysis of the accuracy of distance measurements to fixed points; thus

$$\sigma^2=\sigma_r^2+\dot{r}^2\sigma_T^2. \tag{39}$$

The value of $\dot{r}$ can be easily obtained if we have continuous observations.

Optical measurement of directions.—Measurements by the optical method can be referred either to a system of local coordinates (for example, azimuth and zenith distance) or to a system of universal coordinates, such as apparent right ascension and declination (see p. 108 ff.).

Directions may be obtained from photographs taken with a calibrated camera. With the methods of terrestrial photogrammetry we can use a camera with known elements of exterior orientation. From the picture we can get the direction of a photographed object analytically by measuring the photographic plates with a comparator or directly by using a photogoniometer.

The directions so obtained will refer to the horizontal system $\mathfrak{h}^t$ of the station and thus will be affected by the deflection of the vertical. We can get directions in a general system if we photograph the object with the stars as background. With the help of the stars we will find the apparent right ascension and declination, i.e., the directions in the sidereal system $\mathfrak{Z}$ of coordinates. Those directions can be transformed to any other system with the equations given on page 108 ff. The determination of apparent positions with the help of photography is a common method in astronomy; a summary of the techniques can be found in Smart (1956, chapter 12).

For geodetic applications, the observed object will be moving with respect to the stars, and the timing of the exposure is highly important. Since the stars and (usually) the object itself will not be very bright, the exposure will vary from a fraction of a second to some seconds, depending on the camera. On the other hand, we must know the timing of the exposure to within about 1 millisecond, depending on the apparent velocity of the moving object.

Various methods can be used.

a. The camera is equatorially mounted and follows the stars; thus the image of the object will be a trail. With the help of a special shutter, the trail will be interrupted and the timing of the interruptions will be taken.

b. The camera follows the stars, and the relative motion of the object is compensated with a rotating plate. The time is taken when the rotating plate is in the normal position; this method is used by Markowitz (1954).

c. The camera is kept fixed so that both the stars and the object produce trails as images. The trails will be interrupted for timing with a special shutter.

d. The camera follows the stars while the object (invisible) produce flashes at known times and preferably at equal intervals. Then the plate will contain point images of the stars as well as point images of the flashes. This method has been proposed by Väisälä (1946) and by Atkinson (in 1957).

Methods (a) and (c) can always be used, provided the trail of the object is bright enough to be photographed. Method (b) requires that the relative motion of the object be known. But even if the motion is not completely known, the method will be a help in photographing fainter objects, since the trail will be much smaller. Method (d) applies only to a specially instrumented object.

Figure 6 gives a schematic version of plates taken with the different methods.

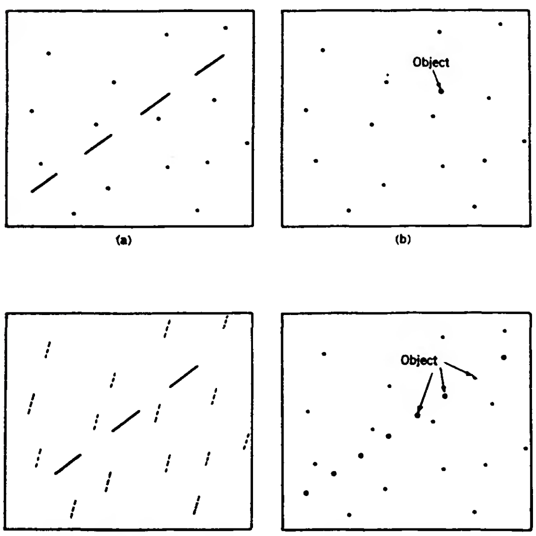

FIGURE 6.—Schematic version of moving objects photographed with four methods.

Alternative methods exist. Instead of using a rotating plate to integrate the light of the object and thus increase the effectiveness of the optics, the camera can follow the object. This principle is used in the Baker-Nunn satellite tracking camera (Henize, 1957). Also, for a flashing object the camera may be kept fixed; the stars will then produce trails and the object will give point images. If a shutter is used to interrupt the trail (of the object or the stars), a correction must be applied either to the positions or to the time, to allow for the fact that the shutter has a finite speed.

Electronic measurement of directions.—Electronic methods using the principle of interference have also been developed to determine the direction of an object which has a transmitter. The Minitrack system (Easton, 1957) is perhaps the best of this type. Such methods give the directions with respect to a local system of antennas, specially arranged and fixed on the ground. By careful and repeated calibrations, however, they can give directions in a general system such as the terrestrial or even the sidereal system.

The accuracy of these electronic methods, at present, is low. Minitrack can provide directions accurate to within one or two minutes of arc, which is low for geodetic applications. Electronic direction methods probably cannot be used, but observations made with electronic systems could give valuable information regarding the variation of the orbital elements, and as such they can be used as described on page 140.

Since conventional radar will give at best an accuracy of $\pm 0^{\circ}.1$, its use is out of the question. A possible method using electronic optical observations is discussed by Merrill (1956).

Correlations between the observed quantities.—If the measurements on the photographic plate were made with equal accuracy to every direction and there were no errors of timing, the values of $\alpha' \cos \delta'$ and δ' would be determined with the same weight. However, the position of the image of the object on the plate is not likely to be determined with the same accuracy in the direction of the apparent motion as in the perpendicular direction. This is more obvious in the case of the methods (a) and (c) on page 112, because the image is in the form of an interrupted line.

Also, we must introduce the effect of errors in timing. Since time is a parameter, we will introduce the error of the timing in the positions observed, but such an error will affect the accuracy only in the direction of the orbit.

We introduce a coordinate system χ and ψ on the plate oriented along the orbit (fig. 7) which

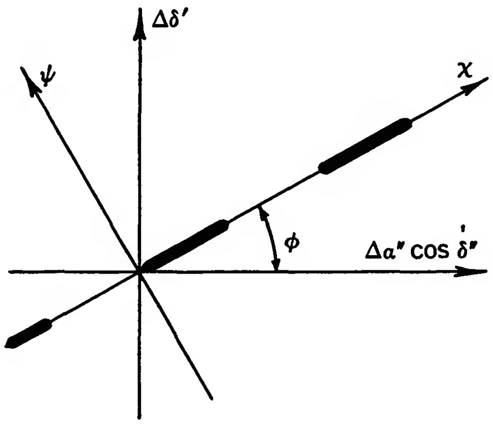

FIGURE 7.—Orientation of the coordinate systems χ and ψ along the orbit.

makes an angle φ with the axis of R.A. We assume that the standard error in the direction of χ is k times the standard error σ in the perpendicular direction ψ. The value of k will depend on the image and on the standard error of timing. It can be evaluated as follows.

If because of the shape of the image alone the standard error in the direction of the orbit is ρ times the standard error in the perpendicular direction, and the standard error in timing is σ_T, we will have

$$k^2\sigma^2=\rho^2\sigma^2+\dot{\chi}^2\sigma_T^2$$

or

$$k^2=\rho^2+\dot{\chi}^2\frac{\sigma_T^2}{\sigma^2}, \tag{40}$$

where $\dot{\chi}$ is the change of χ per unit of time ($\dot{\chi}$ can be evaluated from the plate directly).

If the image of the object is a point, ρ will be 1. Otherwise, it must be determined experimentally, or taken as 1.

The axes χ and ψ being the primary axes of the error ellipse, we have

$$V\left\{\begin{pmatrix}\chi\\ \psi\end{pmatrix}\right\}=\begin{pmatrix}k^2\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}, \tag{41}$$

and since

$$\begin{pmatrix}\Delta\delta'\\ \Delta\alpha'\cos\delta'\end{pmatrix}=\begin{pmatrix}\sin\varphi & \cos\varphi\\ \cos\varphi & -\sin\varphi\end{pmatrix}\begin{pmatrix}\chi\\ \psi\end{pmatrix}=M\begin{pmatrix}\chi\\ \psi\end{pmatrix},$$

therefore

$$V\left\{\begin{pmatrix}\delta'\\ \alpha'\cos\delta'\end{pmatrix}\right\}=MV\left\{\begin{pmatrix}\chi\\ \psi\end{pmatrix}\right\}M',$$

or as shown in equations (42) and (43).

We have found the variance of $\alpha'\cos\delta'$ and δ' as well as the variance of r for the effect of the errors in timing. To complete this analysis of the correlations, we must find the variance of $\alpha'\cos\delta'$, δ', and r as a whole, since all three elements are correlated.

We begin with the fact that if we have a number of reasons to introduce errors to a vector with the partial variances V_1, V_2, . . ., the total variance of the vector will be the sum of the partial variances. Let $\Delta\alpha'\cos\delta'$, $\Delta\delta'$, and Δr be a system of coordinates having its origin at the object, and parallel to the directions of the

$$V\left\{\begin{pmatrix}\delta'\\ \alpha'\cos\delta'\end{pmatrix}\right\}=\sigma^2\begin{bmatrix}k^2\sin^2\varphi+\cos^2\varphi & (k^2-1)\sin\varphi\cos\varphi\\ (k^2-1)\sin\varphi\cos\varphi & k^2\cos^2\varphi+\sin^2\varphi\end{bmatrix}; \tag{42}$$

and since $P=V^{-1}$,

$$P\left\{\begin{pmatrix}\delta'\\ \alpha'\cos\delta'\end{pmatrix}\right\}=\frac{1}{\sigma^2}\begin{bmatrix}\cos^2\varphi+\frac{1}{k^2}\sin^2\varphi & \left(1-\frac{1}{k^2}\right)\sin\varphi\cos\varphi\\ \left(1-\frac{1}{k^2}\right)\sin\varphi\cos\varphi & \sin^2\varphi+\frac{1}{k^2}\cos^2\varphi\end{bmatrix}. \tag{43}$$

apparent R.A., declination, and radius vector (fig. 8).

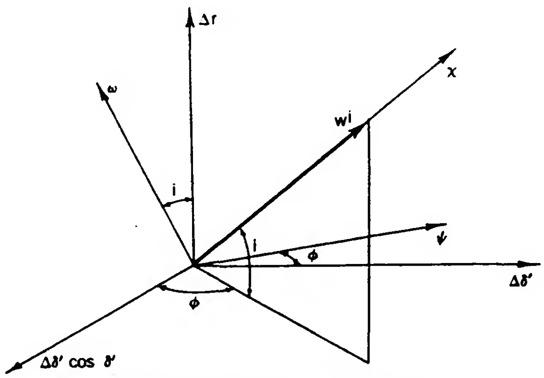

FIGURE 8.—Orientation of the velocity vector.

Let σ be the standard error in the measurement of the directions δ' and $\alpha' \cos \delta'$ (assumed to be equal) and σ_r the standard error of the distance measurements. If the object were not moving, the variance of δ', $\alpha' \cos \delta'$, r would be

$$V_1\left\{\begin{pmatrix} \delta' \\ \alpha' \cos \delta' \\ r \end{pmatrix}\right\}=\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix}. \quad (44)$$

This represents an ellipsoid of revolution.

Further, let w be the relative velocity of the object with respect to the observer, the direction of the velocity being defined in our coordinate system with the angles φ and i (fig. 8).

An error in the timing will introduce new errors in the position. As has already been pointed out, the errors in timing are introduced in the observations, since time is kept as a parameter. The errors thus introduced will be $w\sigma_T$ in the direction of the velocity. They will represent an error ellipsoid that has become a segment of a straight line.

If we introduce the system of coordinates χ, ψ, ω as in figure 8 (the axis χ to the direction of w), the variance in χ, ψ, ω will be

$$V_2\left\{\begin{pmatrix} \chi \\ \psi \\ \omega \end{pmatrix}\right\}=\begin{pmatrix} w^2\sigma_T^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (45)$$

but equation (45a) shows that the variance V_2 expressed in δ', $\alpha' \cos \delta'$, r will be

$$V_2\left\{\begin{pmatrix} \delta' \\ \alpha' \cos \delta' \\ r \end{pmatrix}\right\}=EV_2\left\{\begin{pmatrix} \chi \\ \psi \\ \omega \end{pmatrix}\right\}E'.$$

Or, the variance V_2 will be that shown in equation (46), and the total variance will be

$$V=V_1+V_2,$$

or that of equation (47).

$$\begin{pmatrix} \Delta\delta' \\ \Delta\alpha' \cos \delta' \\ \Delta r \end{pmatrix}=\begin{pmatrix} \dfrac{\sin \varphi \cos i}{r} & \dfrac{\cos \varphi}{r} & \dfrac{-\sin i \sin \varphi}{r} \\ \dfrac{\cos \varphi \cos i}{r} & \dfrac{-\sin \varphi}{r} & \dfrac{-\sin i \cos \varphi}{r} \\ \sin i & 0 & \cos i \end{pmatrix}\begin{pmatrix} \chi \\ \psi \\ \omega \end{pmatrix}=E\begin{pmatrix} \chi \\ \psi \\ \omega \end{pmatrix}. \quad (45a)$$

$$V_2=\sigma_T^2 w^2\begin{pmatrix} \dfrac{1}{r^2} \sin^2 \varphi \cos^2 i & \dfrac{1}{r^2} \sin \varphi \cos^2 i \sin \varphi & \dfrac{1}{r} \sin \varphi \cos i \sin i \\ \dfrac{1}{r^2} \cos \varphi \cos^2 i \sin \varphi & \dfrac{1}{r^2} \cos^2 \varphi \cos^2 i & \dfrac{1}{r} \cos \varphi \cos i \sin i \\ \dfrac{1}{r} \sin \varphi \cos i \sin i & \dfrac{1}{r} \cos \varphi \cos i \sin i & \sin^2 i \end{pmatrix}. \quad (46)$$

$$V=\begin{pmatrix} \sigma^2+\sigma_T^2 \dfrac{w^2}{r^2} \sin^2 \varphi \cos^2 i & \sigma_T^2 \dfrac{w^2}{r^2} \sin \varphi \cos \varphi \cos^2 i & \sigma_T^2 \dfrac{w^2}{r^2} \sin \varphi \cos i \sin i \\ \sigma_T^2 \dfrac{w^2}{r^2} \cos \varphi \sin \varphi \cos^2 i & \sigma^2+\sigma_T^2 \dfrac{w^2}{r^2} \cos^2 \varphi \cos^2 i & \sigma_T^2 \dfrac{w^2}{r^2} \cos \varphi \cos i \sin i \\ \sigma_T^2 \dfrac{w^2}{r^2} \sin \varphi \cos i \sin i & \sigma_T^2 \dfrac{w^2}{r^2} \cos \varphi \cos i \sin i & \sigma_r^2+\sigma_T^2 w^2 \sin^2 i \end{pmatrix}. \quad (47)$$

Introducing the rate of change $\dot{\delta}'$, $\dot{\alpha}' \cos \delta'$, and $\dot{r}$, we obtain the variance as written in equation (48).

The weight matrix is obtained as

$$P = V^{-1}.$$

We may make the following comments on these methods:

Equations (39) and (42) are special cases of the general equation (47).

If the standard errors in R.A. and declination are not equal, this fact can easily be taken into account.

From equation (47) we have eliminated the effect of the image (coefficient ρ), but this effect can also be taken into account.

Similarly, we could also introduce the effect of errors from refraction, since they will also introduce more correlations. However, this effect will not be significant from a practical point of view.

Reduction of observations to the station.—Sometimes the instruments used to measure directions or distances are not centered at the station but are placed eccentrically (in general not far from the station). Since different instruments are needed to measure directions and distances, it will be impossible to have both instruments at the same point. We will, therefore, have to reduce the observations made at the eccentric station. The reduction will be given by a differential formula (see p. 114 ff.).

If we call δ'_e, α'_e, r_e the values observed at the eccentric station with coordinates X^i_e, and δ', α', r are the observations that would have been made from the true station with coordinates X^i_Q, then we will have the formula shown in equation (49).

Correction for aberration.—Usually the object will be moving with respect to the observer and its velocity will be rather high (the velocity of the artificial satellites is of the order of 8 km/sec). This relative motion will introduce an aberration effect which will apparently displace the ray of light.

We could correct for the effect of aberration by the method used in astronomy for observations of planets and comets (planetary aberration), and apply a correction to the time of the observations instead of correcting the observed directions. However, if we wish to keep the time of the observation uncorrected, we must apply the corrections to the observed quantities δ' and α'. In this case, let Q and S be the positions of the station and the object (fig. 9); let $V^i_{\oplus}$ be the velocity of the earth with respect to the sun, V^i_p be the velocity of the observer with respect to the earth, V^i be the velocity of the object with respect to the earth, and V^i_S be its total velocity, $V^i_{\oplus} + V^i$.

The velocity of the object with respect to the observer will then be

$$w^i = V^i_{\oplus} + V^i - V^i_{\oplus} - V^i_p = V^i - V^i_p.$$

The total correction for the aberration will be ϵ (fig. 9) and will have the value

$$\epsilon = \frac{w \cos i}{c}, \tag{50}$$

$w \cos i$ being the projection of w on the plane perpendicular to QS, and c the velocity of light. It will lie on the plane defined by w^i and Q.

We introduce the system of coordinates $\Delta\alpha' \cos \delta'$, $\Delta\delta'$, Δr to the point S, the same

$$V = \begin{bmatrix} \sigma^2 + \sigma_T^2(\dot{\delta}')^2 & \sigma_T^2 \dot{\delta}'(\dot{\alpha}' \cos \delta') & \sigma_T^2 \dot{\delta}' \dot{r} \\ \sigma_T^2 \dot{\delta}'(\dot{\alpha}' \cos \delta') & \sigma^2 + \sigma_T^2(\dot{\alpha}' \cos \delta')^2 & \sigma_T^2(\dot{\alpha}' \cos \delta')\dot{r} \\ \sigma_T^2 \dot{\delta}' \dot{r} & \sigma_T^2(\dot{\alpha}' \cos \delta')\dot{r} & \sigma_r^2 + \sigma_T^2(\dot{r})^2 \end{bmatrix}. \tag{48}$$

$$\begin{bmatrix} \delta' \\ \alpha' \\ r \end{bmatrix} = \begin{bmatrix} \delta'_e \\ \alpha'_e \\ r_e \end{bmatrix} + \begin{bmatrix} -\dfrac{\cos \theta'_e \sin \delta'_e}{r_e} & -\dfrac{\sin \theta'_e \sin \delta'_e}{r_e} & \dfrac{\cos \delta'_e}{r} \\ -\dfrac{\sin \theta'_e}{r_e \cos \delta'_e} & \dfrac{\cos \theta'_e}{r_e \cos \delta'_e} & 0 \\ \cos \theta'_e \cos \delta'_e & \sin \theta'_e \cos \delta'_e & \sin \delta'_e \end{bmatrix} \begin{bmatrix} X^1_e - X^1_Q \\ X^2_e - X^2_Q \\ X^3_e - X^3_Q \end{bmatrix}. \tag{49}$$

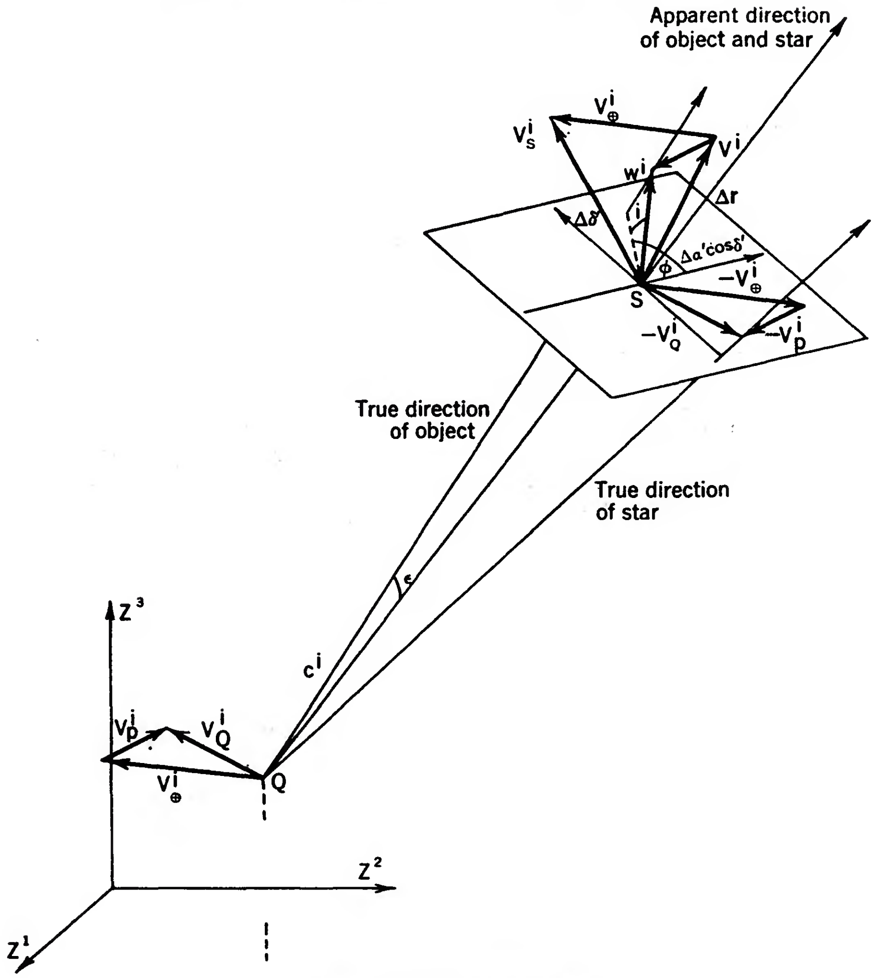

FIGURE 9.—Correction for aberration.

system given on page 114 ff. If w_α, w_δ, w_r are the components of w^i in the three new axes, then

$$w_\alpha = w \cos i \cos \phi$$

$$w_\delta = w \cos i \sin \phi$$

$$w_r = w \sin i.$$

Then the corrections ϵ to α', δ', r will be

$$\epsilon_\alpha = \frac{w_\alpha}{c} \cdot \frac{1}{\cos \delta}$$

$$\epsilon_\delta = \frac{w_\delta}{c} \qquad (51)$$

$$\epsilon_r = \frac{w_r}{c} r.$$

(See footnote.[3])

In the case of photographic determination of the directions, it will be easy to find the values of w_α and w_δ by measuring on the plate the change $\dot{\alpha}'$ and $\dot{\delta}'$ of α' and δ' with time. We have then

$$\epsilon_\alpha = \dot{\alpha} \frac{r}{c \cos \delta}$$

$$(52)$$

$$\epsilon_\delta = \dot{\delta} \frac{r}{c}.$$

[3] Some explanation must be given for this formula. It may seem ridiculous to apply a correction for the velocity of light to the measurement of distances. Actually, no measurement can be made instantaneously, and c should be the . . . velocity of the used yardstick! Since the distances will be measured with electromagnetic waves we must use the velocity of light. Notice, also, that if for the time of the observation we take the mean time between the emission and the reception of the signal, the correction will be zero.

When the directions are obtained by the use of stars, as described on page 111, we will get the apparent direction of the object if the positions of the stars are also apparent. We then apply the corrections given by equation (51) to find the true position (or true direction) of the object. It will be more convenient, however, and will avoid many computations if we use the true rather than the apparent positions of the stars. In this case (fig. 9) we must add (or subtract the negative of) the effect of the aberration of the stars, i.e., the effect of the motion of the observer with respect to the sun (annual and diurnal aberration). This means that we must use V_S^t instead of w^t. Thus we compute the position of the object by using the true positions of the stars; we then apply the correction for the annual aberration for the position of the object as well as the diurnal aberration, and finally apply the correction for the aberration of the object according to equation (51).

Since the diurnal aberration is less than 0ˢ.3, it will in most cases be negligible. The annual aberration can be computed by Bessel's method:

$$Cc+Dd \text{ in R.A.}$$

$$Cc'+Dd' \text{ in declination.}$$

We could further simplify the computations if instead of using the true position of the stars we use the mean position for a given epoch, but the positions of the stars should be corrected for their proper motions.

The position of the object, thus determined, and corrected for the annual aberration as well as for the aberration of its own velocity, will be the mean position for the given epoch, and thus will refer to the **W** system of coordinates. We can then compute the true position of the object by using known methods of reduction, such as Bessel's method, if we want our directions to refer to the **Z** system rather than the **W**.

Correction for optical refraction.—Although the observed objects will be at rather high altitudes, the optical rays will have to pass through the atmosphere and thus undergo a curvature. The existing formulas for the corrections for astronomic and geodetic refraction cannot be applied in our case, because they assume that the observed object is either at infinity or at a rather small distance and low altitude.

We will therefore try to find a formula that can be applied for the intermediate case. We will assume that the density of the atmosphere ρ as a function of altitude H is given by the relation,

$$\log \rho = K_1 H + K_2. \tag{53}$$

The observed values of ρ at different altitudes H (Whipple, 1954) fit accurately enough to those of equation (53) for heights up to 80 to 100 km. The results from the artificial satellites for the density at heights of 200 to 600 km show an important deviation from the values of equation (53) (Whitney, 1959), but this does not affect our solution (see p. 118).

The coefficients K_1 and K_2 will be determined experimentally. Or we can write

$$\rho = e^{aH+b},$$

where $a=\frac{K_1}{M}$, $b=\frac{K_2}{M}$ $\qquad (M=\log_{10} e)$.

But if n is the index of refraction, $(n-1)=B\rho$ according to Gladstone's law, and so

$$(n-1)=Be^{aH+b}$$

or

$$(n-1)=ke^{aH}, \tag{54}$$

where $k=Be^b=n_0-1$, if n_0 is n for $H=0$.

We assume that the index of refraction as a function of the altitude H is given by equation (54), k and a being coefficients whose values will be determined experimentally.

Let Q (fig. 10) be the station and S the observed object. The ray of light will follow the curved line SMQ, and the observed zenith distance will be Z_0, the angle between the vertical and the tangent to the curve at Q. Also let ϵ be the angle between the tangent at any point M with the tangent at Q.

Assuming no curvature of the earth (this assumption will hold for $Z<45°$), we have the relation

$$d\epsilon = -\tan Z \frac{dn}{n}, \tag{55}$$

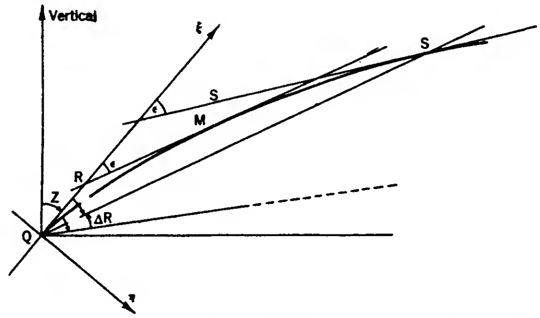

FIGURE 10.—Correction for refraction, when curvature of the earth is not considered.

which, since Z will be constant, will give by integration

$$\epsilon = \tan Z_0(\ln n_0 - \ln n). \qquad (56)$$

Since n is very near 1, we can expand the equation so that

$$\epsilon = \tan Z_0[(n_0-1)-(n-1)]. \qquad (57)$$

If we introduce a system of coordinates ξ and η so that ξ is the tangent at Q we have

$$\frac{d\eta}{d\xi} = \epsilon = \tan Z_0[(n_0-1)-(n-1)] \qquad (58)$$

or with the help of equation (54)

$$\frac{d\eta}{d\xi} = \tan Z_0(k - ke^{aH}).$$

But since the angle ϵ is small, we can replace ξ with r (r is the distance QS); thus $H = r \cos Z$. Then equation (58) is written

$$\frac{d\eta}{d\xi} = k \tan Z_0 - k \tan Z_0 e^{a\xi \cos Z_0}, \qquad (59)$$

or

$$\eta = \int_0^\xi (k \tan Z_0 - k \tan Z_0 e^{a\xi \cos Z_0})\, d\xi.$$

Integrating, we obtain

$$\eta = k \tan Z_0 \xi - \frac{k \tan Z_0}{a \cos Z_0} e^{a \cos Z_0 \xi} + C.$$

For $\xi = 0$, $\eta = 0$; therefore

$$C = \frac{k \tan Z_0}{a \cos Z_0}.$$

Thus we obtain an equation that expresses the line followed by the ray of light:

$$\eta = k \tan Z_0 \xi + \frac{k \tan Z_0}{a \cos Z_0}(1 - e^{a \cos Z_0 \xi}). \qquad (60)$$

If R is the correction, we have $\tan R = \frac{\eta}{\xi} \simeq R$ since R is small, and if we replace ξ with r we get an equation which gives the correction for the refraction R as a function of the zenith distance Z_0 and distance r:

$$R = k \tan Z_0 + \frac{k \tan Z_0}{a \cos Z_0 r}(1 - e^{a \cos Z_0 r}). \qquad (61)$$

This is a general formula and can be applied for any r provided $Z < 45°$. We can find that for $r = \infty$ (we have $a < 0$), the correction is $k \tan Z_0$, which is the well-known astronomic refraction.

For the constants, we take $k = (n_0 - 1) = 292 \times 10^{-6}$ or 60ʺ25 for the normal case (0°, 760 mm Hg), while for K_1 of equation (53) we have estimated from the work of Whipple (1954) that $K_1 = -0.06$ km^{-1}, which brings $a = -0.1385$ km^{-1}.

Then for the normal case we get

$$R = 60''25 \tan Z_0 - 435''0 \frac{\tan Z_0}{\cos Z_0 r}(1 - e^{-0.1385 \cos Z_0 r}), \qquad (62)$$

where r is expressed in km.

From equation (57) we see that for an observation made at a height between 100 km and infinity, the angle ϵ will be

$$\epsilon_{100}^{\infty} = \tan Z(n_{100} - 1),$$

and since $(n_{100} - 1) \simeq 0.5 \times 10^{-9} = 0''0001$,

$$\epsilon_{100}^{\infty} = 0''0001 \tan Z,$$

which proves that even for values of $Z = 80°$, the angle ϵ_{100}^{∞} will be less than 0ʺ001. This means that we need not consider the atmosphere above a height of 100 km, since the errors introduced by this neglect will be very much smaller than the errors of our measurements.

As follows from equation (60), the curve of the ray of light has an asymptote (fig. 11), given by the expression,

$$\eta = (k \tan Z_0)\xi + \left(\frac{k \tan Z_0}{a \cos Z_0}\right). \qquad (63)$$

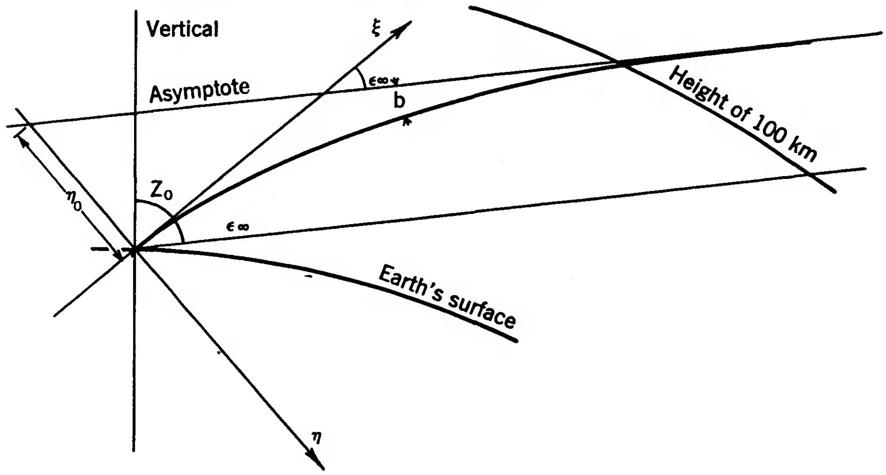

FIGURE 11.—Asymptote of a refracted ray of light.

The curve will fast approach the asymptote. With the assumed constants, the distance between the curve and the asymptote will be

$$b=\frac{0.0021 \tan Z_0}{\cos Z_0} e^{-0.1385 \cos Z_0 \xi} \quad \text{(in km)}. \quad (64)$$

For positions of the object not far from Q the elimination of the curvature of the earth (and thus the curvature of the equal n layers) will not have any important effect. This will be true for points within $Z \simeq 45°$ and $H=100$ km. In such a case, confusing H with $r \cos Z$ will not have important effects. For points beyond this limit, we cannot replace H with $r \cos Z$. But provided $Z<45°$, the curve will almost have completed its shape when it reaches the height of 100 km. In such an extreme case, the value of b will be about 1 mm.

Actually, in this case, we could use the equation of the asymptote, and from it compute the correction,

$$k \tan Z_0 + \frac{k \tan Z_0}{a \cos Z_0 r}.$$

We prefer to leave the formula for the correction as given in equation (61) since for points at distances farther than our limit, the term $(1-e^{a \cos Z_0 r})$ will differ from unity by only 10^{-9}. Thus equation (61) holds also for every value of r, provided $Z<45°$.

If the directions were obtained photographically with reference to stars, as described on page 111 ff. the correction should be $\Delta R=\epsilon_\infty - R_S$ or $\Delta R=R_\infty - R_S$. We then obtain

$$\Delta R=-435\overset{''}{.}0 \frac{\tan Z_0}{\cos Z_0 r}(1-e^{-0.1385 \cos Z_0 r}). \quad (65)$$

A nomogram (fig. 22) is given on page 158 for this correction, which is always negative.

This correction will be in the zenith distance. The correction in α' and δ' will be given by the formulas

$$C_\alpha = -\Delta R \sec \delta' \sin q'$$

$$C_\delta = -\Delta R \cos q'$$

where q' is the apparent parallactic angle of the object.

We may want also to consider the effect of the earth's curvature. Since layers of equal n have a curvature following the curvature of the earth, the value of Z in equation (55) will not be constant, and equation (56) will no longer be its integral. The angle Z (fig. 12) will be

$$Z=Z_0-\omega,$$

and since the height of the atmosphere has been taken as 100 km, ω will be a small angle so that

$$\omega=\frac{H}{\rho}\tan Z_0,$$

where ρ is the radius of curvature of the earth

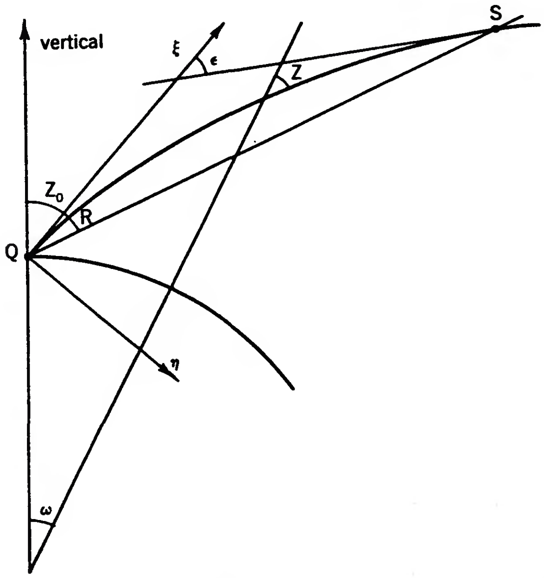

FIGURE 12.—Correction for refraction, including the effect of the earth's curvature.

Then

$$\tan Z=\tan Z_0-\frac{H\tan Z_0}{\rho\cos^2 Z_0}. \tag{66}$$

Thus equation (55) will be

$$d\epsilon=-\tan Z_0\frac{dn}{n}+\frac{\tan Z_0}{\rho\cos^2 Z_0}H\frac{dn}{n}. \tag{67}$$

Integrating, we obtain

$$\epsilon=\tan Z_0(k-ke^{aH})+\frac{\tan Z_0}{\rho\cos^2 Z_0}\int_{n_0}^{n}H\frac{dn}{n}$$

$$=\tan Z_0(k-ke^{aH})+\frac{\tan Z_0}{\rho\cos^2 Z_0}\int_0^H Hake^{aH}dH$$

$$=\tan Z_0(k-ke^{aH})+\frac{ak\tan Z_0}{\rho\cos^2 Z_0}\left(\frac{e^{aH}}{a^2}(aH-1)+\frac{1}{a^2}\right).$$

But $\epsilon=\frac{d\eta}{d\xi}$ so that

$$\eta=k\tan Z_0\int d\xi-k\tan Z_0\int e^{aH}\left(\frac{d\xi}{dH}\right)dH+$$
$$\frac{k\tan Z_0}{\rho\cos^2 Z_0}\int He^{aH}\left(\frac{d\xi}{dH}\right)dH-$$
$$\frac{k\tan Z_0}{a\rho\cos^2 Z_0}\int e^{aH}\left(\frac{d\xi}{dH}\right)dH+\frac{k\tan Z_0}{a\rho\cos^2 Z_0}\int d\xi. \tag{68}$$

Also

$$\frac{d\xi}{dH}=\frac{1}{\cos Z},$$

and since

$$\cos Z=\cos Z_0(1+\frac{\tan^2 Z_0}{\rho}H),$$

$$\frac{d\xi}{dH}=\frac{1}{\cos Z_0}-\frac{\tan^2 Z_0}{\rho\cos Z_0}H.$$

So equation (68) is written as in equation (68a).

$$\eta=k\tan Z_0\int d\xi-\frac{k\tan Z_0}{\cos Z_0}\int e^{aH}dH+\frac{k\tan^3 Z_0}{\rho\cos Z_0}\int He^{aH}dH+\frac{k\tan Z_0}{\rho\cos^3 Z_0}\int Ha^{aH}dH-$$
$$\frac{k\tan^3 Z_0}{\rho^2\cos^3 Z_0}\int H^2e^{aH}dH-\frac{k\tan Z_0}{a\rho\cos^3 Z_0}\int e^{aH}dH+\frac{k\tan^3 Z_0}{a\rho^2\cos^2 Z_0}\int He^{aH}dH+\frac{k\tan Z_0}{a\rho\cos^2 Z_0}\int d\xi. \tag{68a}$$

Integrating and eliminating second order terms $\left(\text{terms containing } \frac{H^2}{\rho^2}\right)$, we obtain

$$\eta = k \tan Z_0 \xi - \left(\frac{k \tan Z_0}{a \cos Z_0} + \frac{k \tan Z_0}{a^2 \rho \cos^3 Z_0}\right) e^{aH} + \left(\frac{k \tan^3 Z_0}{a^2 \rho \cos Z_0} + \frac{k \tan Z_0}{a^2 \rho \cos^3 Z_0}\right) e^{aH}(aH-1) + \frac{k \tan Z_0}{a\rho \cos^2 Z_0} \xi + C.$$

The constant of integration is found from the condition that at the origin $\eta = 0$. We get finally equation (69) and thus, since $R = \frac{\eta}{\xi}$, we obtain equation (70).

Introducing the values of the constants, we get equation (71) which gives the correction for the refraction for values of Z up to perhaps 80°. The last term usually can be neglected.

(All formulas given in this section are based on normal conditions for the value of k. If the temperature and pressure at the station are different, k should be changed. A nomogram (fig. 23) is given on page 159 for this correction. The height of the station H_0 will not change the results except for extremely high altitudes. In such a case the integration must begin from H_0.)

It can be seen that for $Z_0 < 45°$, equation (71) gives practically the same correction as equation (62).

Formulas (69) to (71) will be correct for $Z_0 < 75°$ (or perhaps 80°) and $H < 100$ km since we have expanded Z in series and eliminated second order terms. For $H > 100$ km, as we have shown, the path of the ray will, practically, already have taken its shape and will follow the asymptote. The equation of the asymptote will be

$$\eta = \left(k \tan Z_0 + \frac{k}{a\rho} \tan Z_0 \sec^2 Z_0\right) \xi + \frac{k}{a} \tan Z_0 \sec Z_0 \left[1 + \frac{1}{a\rho}(2 \sec^2 Z_0 + \tan^2 Z_0)\right]. \quad (72)$$

Thus the correction will be:

$$R = k \tan Z_0 + \frac{k}{a\rho} \tan Z_0 \sec^2 Z_0 + \frac{k}{ar} \tan Z_0 \sec Z_0 \left[1 + \frac{k}{a\rho}(2 \sec^2 Z_0 + \tan^2 Z_0)\right]. \quad (73)$$

But as with equation (61), we leave the formula for the correction as given in equation (71) since the additional terms will bring practically no change.

Thus we can check the fact that for points at infinity we get

$$R = 60\overset{''}{.}25 \tan Z_0 - 0\overset{''}{.}0682 \tan Z_0 \sec^2 Z_0$$

or

$$R = 60\overset{''}{.}18 \tan Z_0 - 0\overset{''}{.}0682 \tan^3 Z_0,$$

which is exactly the general formula of astronomic refraction.

$$\eta = (k \tan Z_0)\xi + \left[\frac{k}{a\rho} \tan Z_0 \sec^2 Z_0\right]\xi + \frac{k}{a} \tan Z_0 \sec Z_0 \left[1 + \frac{1}{a\rho}(2 \sec^2 Z_0 + \tan^2 Z_0)\right](1 - e^{aH}) + \frac{k}{a\rho} \tan Z_0 \sec Z_0 (\sec^2 Z_0 + \tan^2 Z_0) H e^{aH}. \quad (69)$$

$$R = k \tan Z_0 + \frac{k}{a\rho} \tan Z_0 \sec^2 Z_0 + \frac{k}{ar} \tan Z_0 \sec Z_0 \left[1 + \frac{1}{a\rho}(2 \sec^2 Z_0 + \tan^2 Z_0)\right](1 - e^{aH}) + \frac{k}{a\rho r} \tan Z_0 \sec Z_0 (\sec^2 Z_0 + \tan^2 Z_0) H e^{aH}. \quad (70)$$

$$R = 60\overset{''}{.}25 \tan Z_0 - 0\overset{''}{.}0682 \tan Z_0 \sec^2 Z_0 - 435\overset{''}{.}0 \tan Z_0 \sec Z_0 [1 - 0.00113(2 \sec^2 Z_0 + \tan^2 Z_0)](1 - e^{-0.1385H}) - 0\overset{''}{.}0682 \tan Z_0 \sec Z_0 (\sec^2 Z_0 + \tan^2 Z_0) \frac{H}{r} e^{-0.1385H}. \quad (71)$$

When the directions are obtained photographically with the stars as reference (p. 111), the correction will be that given in equation (74). The last term generally is negligible. Also, since the term 0.00113 ($\tan^2 Z_0+2\sec^2 Z_0$) is about 0.04 for $Z_0=75°$, we could say that in most cases equation (65) will be sufficient.

Correction for electronic refraction.—The refraction of electromagnetic waves in the earth's atmosphere is a very complicated problem. It depends on the wave length and also on the dielectric condition of the air. Different ionized layers exist at altitudes that vary with place and time, where reflections and refractions occur. This greatly complicates the problem. However, the shorter the wave length the less is the influence of those layers. The directions thus obtained with radio methods, as described on page 112, must be corrected.

The distances obtained with radio methods need two corrections, the first for the curvature of the path followed by the electromagnetic wave, and the second for the change of the value of the velocity along the path.

These corrections have been studied (Jacobsen, 1951; Williams, 1951) for low altitudes for applications in electronic geodesy (Shoran, etc.), but it is very doubtful whether they can be applied also in the case of high altitudes. With the use of radio methods, however, the distances will probably be more accurately measured than the directions.

Observations to objects of known positions

Several methods exist for making observations to objects of known positions.

Use of angles in space.—Let us consider first the case in which we have observed angles to three known positions. Let S_1, S_2, S_3 (fig. 13) be the known positions of the object, and Q the unknown station where the angles $\beta^{(12)}$, $\beta^{(23)}$, $\beta^{(31)}$ have been observed.

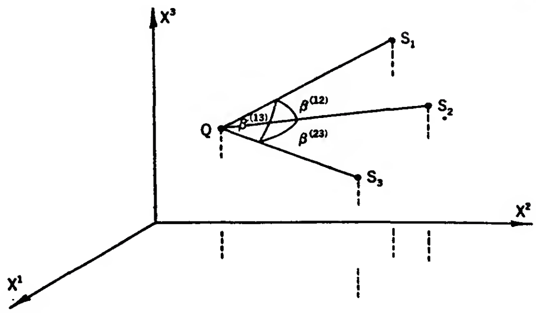

FIGURE 13.—Determining the position of an object by use of three angles in space.

In general there is a solution giving the position of Q, provided S_1, S_2, S_3 are not colinear. The problem is that of resection in space, and Q will lie on the intersection of three tores. The direct solution gives a system of high order and is not applicable. This problem is of great importance in photogrammetry. Of the many solutions suggested, the best, which has also the advantage of being easily generalized in a case of more than three known points S, seems to be one in which we compute corrections dX^i to an approximate position $\overline{X}^i$ of the point Q. This principle is very much used in geodesy.

Briefly, the solution is obtained as follows: Let $\overline{Q}$ be the approximate position of Q. Using $\overline{Q}$ and S_1, S_2, S_3, we compute the angles $\overline{\beta}^{(12)}$, $\overline{\beta}^{(23)}$, $\overline{\beta}^{(31)}$.

If $\overline{Q}$ is near Q and the corrections to $\overline{Q}$ are dX^i we obtain the equation:

$$\frac{\partial \cos \overline{\beta}^{(12)}}{\partial X^1}\, dX^1+\frac{\partial \cos \overline{\beta}^{(12)}}{\partial X^2}\, dX^2+$$

$$\frac{\partial \cos \overline{\beta}^{(12)}}{\partial X^3}\, dX^3=\cos \beta^{(12)}-\cos \overline{\beta}^{(12)} \qquad (75)$$

and two more equations with $\beta^{(23)}$ and $\beta^{(31)}$.

$$R=-435\overset{''}{.}0\,\frac{\tan Z_0 \sec Z_0}{r}[1-0.00113(\tan^2 Z_0+2\sec^2 Z_0)]\,(1-e^{-0.1385H})-$$

$$0\overset{''}{.}0682\,\frac{\tan Z_0 \sec Z_0}{r}(\tan^2 Z_0+\sec^2 Z_0)He^{-0.1385H}. \qquad (74)$$

The coefficients of dX^i are of the form

$$\frac{\partial \cos \beta^{(jk)}}{\partial X^1} = \cos a_j^i \left[\frac{1}{QS_k} - \frac{\cos \beta^{(jk)}}{QS_j}\right] + \cos a_k^i \left[\frac{1}{QS_j} - \frac{\cos \beta^{(jk)}}{QS_k}\right],$$

where a^i is the direction angle of the line SQ.

The solution of this system of three equations with three unknowns gives the correction dX^i to the approximate values $\overline{X}^i$.

The method can be generalized for the case in which we have observed more than three angles $\beta^{(jk)}$. The solution will be obtained by the method of least squares. We must notice, however, that if equation (75) is applicable in the case of three known points, it is not applicable if we must make an adjustment, since $\cos \beta^{(jk)}$ is not the measured quantity. In this case we must use equations with $\beta^{(jk)} - \overline{\beta}^{(jk)}$ since the angles β are assumed to be measured directly.

We must replace

$$\frac{\partial \cos \beta}{\partial X} \text{ with } \frac{\partial \beta}{\partial X} \text{ or } \frac{\partial \cos \beta}{\partial X} \cdot \frac{\partial \beta}{\partial \cos \beta}.$$

We then get equations of the form

$$A_i^{(jk)} dX^i = \beta^{(jk)} - \overline{\beta}^{(jk)}, \tag{76}$$

where

$$A_i^{(jk)} = -\frac{\cos a_j^i}{\sin \beta^{(jk)}} \left[\frac{1}{QS_k} - \frac{\cos \beta^{(jk)}}{QS_j}\right] - \frac{\cos a_k^i}{\sin \beta^{(jk)}} \left[\frac{1}{QS_j} - \frac{\cos \beta^{(jk)}}{QS_k}\right].$$

The accuracy of this method of determining the position of the station depends on the accuracy of the observations and on the net configuration (the positions of the object are assumed to be errorless).

An analysis to estimate the accuracy could be made, but since this method will not be of much use we shall not attempt it. The reader may refer, however, to a similar study for photogrammetric applications (Doyle, 1957). We may note, also, that if the different positions of the object lie near a straight line (as happens with an artificial satellite) the solution is very weak.

Use of directions and distances.—Let Z_Q^i (fig. 14) be the coordinates of the station Q, Z_S^i the coordinates of the object S, r^i the vector QS, and l^i the direction cosines (unit vector) in the Z system of the vector r^i corresponding to α' and δ'.

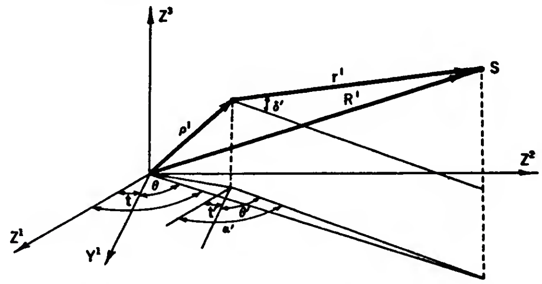

FIGURE 14.—Relation between the positions of the station and the object.

The equation relating the positions of Q and S is

$$R^i = \rho^i + r^i$$

or

$$Z_S^i = Z_Q^i + l^i r. \tag{77}$$

If the coordinates are referred to the terrestrial system, we will have

$$X_S^i = X_Q^i + n^i r. \tag{78}$$

Again, since we almost always know the approximate positions, we will develop formulas relating the corrections to the approximate coordinates $\overline{Z}^i$ or $\overline{X}^i$ and use the method of variation of coordinates.

Differentiating equation (77) we get

$$dZ_S^i = dZ_Q^i + dl^i r + l^i dr, \tag{79}$$

where l^i is a function of α' and δ'. Differentiating equation (29) we get

$$dl^1 = -(\cos \alpha' \sin \delta')d\delta' - (\sin \alpha' \cos \delta')d\alpha',$$

$$dl^2 = -(\sin \alpha' \sin \delta')d\delta' + (\cos \alpha' \cos \delta')d\alpha', \tag{80}$$

$$dl^3 = (\cos \delta')d\delta'.$$

Then equation (79) can be written in the form shown in equations (81).

$$
\begin{aligned}
(dZ^1_S-dZ^1_Q)&=-(\cos\bar{\alpha}'\sin\bar{\delta}')\bar{r}\,d\delta'-\\
&\quad(\sin\bar{\alpha}')\bar{r}\cos\bar{\delta}'\,d\alpha'+(\cos\bar{\alpha}'\cos\bar{\delta}')dr,\\
(dZ^2_S-dZ^2_Q)&=-(\sin\bar{\alpha}'\sin\bar{\delta}')\bar{r}\,d\delta'+\\
&\quad(\cos\bar{\alpha}')\bar{r}\cos\bar{\delta}'\,d\alpha'+(\sin\bar{\alpha}'\cos\bar{\delta}')dr,\\
(dZ^3_S-dZ^3_Q)&=+(\cos\bar{\delta}')\bar{r}\,d\delta'+\\
&\quad 0+(\sin\delta')dr.
\end{aligned}
\tag{81}
$$

Inverting the system (81) (the matrix is orthonormal) we get

$$
\begin{aligned}
d\delta'&=-\left(\frac{\cos\bar{\alpha}'\sin\bar{\delta}'}{r}\right)(dZ^1_S-dZ^1_Q)-\\
&\quad\left(\frac{\sin\bar{\alpha}'\sin\bar{\delta}'}{r}\right)(dZ^2_S-dZ^2_Q)+\\
&\quad\left(\frac{\cos\bar{\delta}'}{r}\right)(dZ^3_S-dZ^3_Q),\\
d\alpha'&=-\left(\frac{\sin\bar{\alpha}'}{r\cos\delta'}\right)(dZ^1_S-dZ^1_Q)+\\
&\quad\left(\frac{\cos\bar{\alpha}'}{r\cos\delta'}\right)(dZ^2_S-dZ^2_Q),\\
dr&=(\cos\bar{\alpha}'\cos\bar{\delta}')(dZ^1_S-dZ^1_Q)+\\
&\quad(\sin\bar{\alpha}'\cos\bar{\delta}')(dZ^2_S-dZ^2_Q)+\\
&\quad(\sin\bar{\delta}')(dZ^3_S-dZ^3_Q).
\end{aligned}
\tag{82}
$$

We will write this system

$$
\begin{pmatrix} d\delta \\ d\alpha \\ dr \end{pmatrix}=A^r_i\,(dZ^i_S-dZ^i_Q).
\tag{82a}
$$

In the **Y** system, assuming that t is known, we replace α' with $\theta'=\alpha'-t$, and Z^i with Y^i and find that

$$
\begin{aligned}
d\delta'&=-\left(\frac{\cos\bar{\theta}'\sin\bar{\delta}'}{\bar{r}}\right)(dY^1_S-dY^1_Q)-\\
&\quad\left(\frac{\sin\bar{\theta}'\sin\bar{\delta}'}{\bar{r}}\right)(dY^2_S-dY^2_Q)+\\
&\quad\left(\frac{\cos\bar{\delta}'}{\bar{r}}\right)(dY^3_S-dY^3_Q),\\
d\theta'=d\alpha'&=-\left(\frac{\sin\bar{\theta}'}{\bar{r}\cos\delta'}\right)(dY^1_S-dY^1_Q)+\\
&\quad\left(\frac{\cos\bar{\theta}'}{\bar{r}\cos\delta'}\right)(dY^2_S-dY^2_Q),\\
dr&=(\cos\bar{\theta}'\cos\bar{\delta}')(dY^1_S-dY^1_Q)+\\
&\quad(\sin\bar{\theta}'\cos\bar{\delta}')(dY^2_S-dY^2_Q)+\\
&\quad(\sin\bar{\delta}')(dY^3_S-dY^3_Q),
\end{aligned}
\tag{83}
$$

or

$$
\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix}=\Theta^r_i\,(dY^i_S-dY^i_Q).
\tag{83a}
$$

Equations (83) or (83a) can also be used for the terrestrial (**X**) and geodetic (**x**) systems, since the omitted terms will be of the order of $dX\times10^{-6}$ (for $dX=1000$ meters we will omit terms of 1 millimeter).

Hence we get

$$
\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix}=\Theta^r_i(dX^i_S-dX^i_Q)=\Theta^r_i(dx^i_S-dx^i_Q)
\tag{84}
$$

with

$$
\Theta^r_i=\begin{pmatrix}
-\dfrac{\cos\bar{\theta}'\sin\bar{\delta}'}{\bar{r}} & -\dfrac{\sin\bar{\theta}'\sin\delta'}{\bar{r}} & \dfrac{\cos\bar{\delta}'}{\bar{r}}\\[2ex]
-\dfrac{\sin\bar{\theta}'}{\bar{r}\cos\bar{\delta}'} & \dfrac{\cos\bar{\theta}'}{\bar{r}\cos\bar{\delta}'} & 0\\[2ex]
\cos\bar{\theta}'\cos\bar{\delta}' & \sin\bar{\theta}'\cos\bar{\delta}' & \sin\bar{\delta}'
\end{pmatrix},
$$

and $d\delta'=\delta'-\bar{\delta}'$, $d\alpha'=\alpha'-\bar{\alpha}'$, $dr=r-\bar{r}$, which are the observed values minus those computed from the approximate coordinates.

The computed values of $\bar{\delta}'$, $\bar{\alpha}'$, (or $\bar{\theta}'$), $\bar{r}$ can be obtained from the rectangular coordinates of the object, with the help of equations (30), (32), (34), or (35), depending on the system in which the coordinates are given. If the position of the object is given in spherical coordinates α, δ, R, in the **Z** system, we can find the apparent $\bar{\alpha}'$, $\bar{\delta}'$, (or $\bar{\theta}'$), $\bar{r}$ as follows (fig. 15):

$$
\begin{aligned}
(\tan\bar{\alpha}'-\alpha)&=\tan(\bar{\theta}'-\theta)=\frac{\bar{Y}^1_Q\sin\theta-\bar{Y}^2_Q\cos\theta}{\bar{A}},\\
\tan\bar{\delta}'&=\frac{\cos(\bar{\theta}'-\theta)[R\sin\delta-\bar{Y}^3_Q]}{\bar{A}},\\
\bar{r}&=\frac{\bar{A}}{\cos(\bar{\theta}-\theta)\cos\bar{\delta}},
\end{aligned}
\tag{85}
$$

where

$$
\bar{A}=R\cos\delta-\bar{Y}^1_Q\cos\theta-\bar{Y}^2_Q\sin\theta.
$$

If from one station we have observed α', δ' and r of an object of known position, we can find the position of the station with the help of equation (84).

If we have observed from the same station many positions j of the object and we have all α'^j, δ'^j, r^j, we adjust by the method of least squares for the coordinates of the station by using the appropriate weights and correlations, if any.

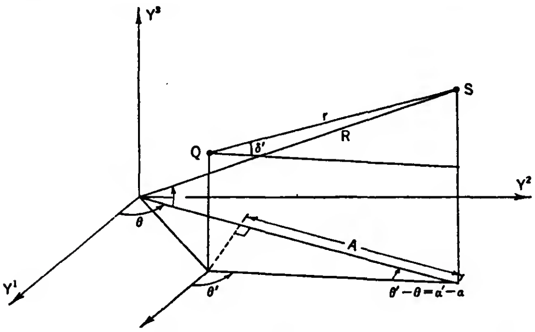

FIGURE 15.—Topocentric and geocentric directions.

It may happen that we do not observe both directions and distances. If we observe only directions, i.e., α'^j and δ'^j, for every observation we will have two equations, the first two of equation (84). If we have measured only the distances r^j, we will have for every observation one equation, the last of equation (84).

We can also get a solution if we have observed only declinations but we do not get a complete solution if we have observed only R.A., since α is independent of X^3.

The same equation (84) can be applied both to problems of resection (i.e., from known positions of the object compute the position of the station) and to problems of intersection (i.e., from known positions of stations compute the position of the object). In the latter case, since the object will be moving, we must make the observations simultaneously from all stations.

If we do not want the rectangular coordinates of the station (or the object) but instead want the ecliptic coordinates φ, λ, H (or δ, α, R), we must replace dX^i_Q with $d\varphi$, $d\lambda$, dH (or dX^i_S with $d\delta$, $d\alpha$, dR). We therefore introduce a system of coordinates with origin at the approximate point $\overline{Q}$. The 3–axis is directed toward the normal (zenith), the 2–axis is directed toward north on the horizontal plane, and the 1–axis toward east.

We will call this system horizontal and we will use the symbol $\mathfrak{h}$. We find that the relation between this and the terrestrial system is given by the formula of equation (86). Or,

$$h^j = H^j_i (X^i - \overline{X}^i),$$

$\overline{X}^i$ and φ, λ being the coordinates of $\overline{Q}$. By using differential displacements, and since $dh^1 = N \cos\varphi\, d\lambda$, $dh^2 = \rho\, d\varphi$, $dh^3 = dH$, we obtain the matrix (87) or (88). Or,

$$h^j = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda\sin\varphi & -\sin\lambda\sin\varphi & \cos\varphi \\ \cos\lambda\cos\varphi & \sin\lambda\cos\varphi & \sin\varphi \end{bmatrix} (X^i - \overline{X}^i), \tag{86}$$

$$\begin{bmatrix} N\cos\varphi\, d\lambda \\ \rho\, d\varphi \\ dH \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda\sin\varphi & -\sin\lambda\sin\varphi & \cos\varphi \\ \cos\lambda\cos\varphi & \sin\lambda\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} dX^1 \\ dX^2 \\ dX^3 \end{bmatrix} \tag{87}$$

$$\begin{bmatrix} dX^1 \\ dX^2 \\ dX^3 \end{bmatrix} = \begin{bmatrix} -N\sin\lambda\cos\varphi & -\rho\cos\lambda\sin\varphi & \cos\lambda\cos\varphi \\ N\cos\lambda\cos\varphi & -\rho\sin\lambda\sin\varphi & \sin\lambda\cos\varphi \\ 0 & \rho\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} d\lambda \\ d\varphi \\ dH \end{bmatrix} \tag{88}$$

$$dX^i = L^i_j \begin{bmatrix} d\lambda \\ d\varphi \\ dH \end{bmatrix}. \qquad (88a)$$

If the height of the station is not negligible, N and ρ should be replaced with $N+H$, and $\rho+H$.

Substituting in equation (84) we get

$$\begin{bmatrix} d\delta' \\ d\theta' \\ dr \end{bmatrix} = \Theta^r_i dX^i_s - \Theta^r_i L^i_j \begin{bmatrix} d\lambda \\ d\varphi \\ dH \end{bmatrix}. \qquad (84a)$$

Similarly, we get the matrix in equation (89). Or,

$$dX^i = S^i_k \begin{bmatrix} d\delta \\ d\alpha \\ dR \end{bmatrix} \qquad (89a)$$

and equation (84) becomes

$$\begin{bmatrix} d\delta' \\ d\alpha' \\ dr \end{bmatrix} = \Theta^r_i S^i_k \begin{bmatrix} d\delta \\ d\alpha \\ dR \end{bmatrix} - \Theta^r_i dX^i_Q. \qquad (84b)$$

Accuracy of the determined positions.—After the solution of the matrix equation (84), or after the adjustment of the same equation, in the case of superfluous observations, the accuracy with which the positions are determined depends on three things: the accuracy of the measured quantities, the accuracy of the given positions, and the net configuration.

Since the problems of resection and intersection are the same (in both cases we have measured the same directions and distances), we shall study the case in which we determine the position of one station Q from a number of known positions j ($j = 1, 2, \ldots s$) of the object (or objects), having measured all or some of the elements α', δ', r.

Both the accuracy of the observations and the net configuration will affect the accuracy of the determined positions. On the assumption that the positions $X^i_{S_j}$ are given and correct, the solution for dX^i_Q (or X^i) will be given by the system

$$A^q_i dX^i = l^q. \qquad (90)$$

Here A^q_i is a matrix consisting of matrices $-\Theta^r_i$ arranged in columns, and l^q is a vector consisting of vectors $(d\delta', d\alpha', dr)$ arranged in a column. If we have observed all three elements at every position j, we have $q=3j$. The solution by least squares will be given by the equation

$$NdX = A'\cdot P\cdot l$$

or

$$dX = N^{-1}A'\cdot P\cdot l, \qquad (91)$$

where $N=A'\cdot P\cdot A$ and P is the weight matrix, or $P=V^{-1}$, V being the variance of the measured quantities l^q.

Furthermore, the variance $V\{X_Q\}$ which is the same as the variance of dX_Q will be

$$V\{X_Q\} = N^{-1} = (A'\cdot P\cdot A)^{-1}. \qquad (92)$$

If the values l^q are not correlated and have the same variance σ_l^2, $P = \frac{1}{\sigma_l^2} I$ and

$$V\{X_Q\} = \sigma_l^2 (A'\cdot A)^{-1}. \qquad (92a)$$

We see from equation (92a) that the variance of X_Q (or in other words the standard error of X_Q) depends on σ_l=the standard error of the observed quantities. There is no doubt that the smaller the value of σ_l, the more accurate point Q will be. But $V\{X_Q\}$ also depends on $(A'\cdot A)^{-1}$ (or in the more general case on N^{-1}) and $(A'\cdot A)^{-1}$ is a function of the geometry of the net, as can easily be proved.

$$\begin{bmatrix} dX^1 \\ dX^2 \\ dX^3 \end{bmatrix} = \begin{bmatrix} -R\cos\theta\sin\delta & -R\sin\theta\cos\delta & \cos\theta\cos\delta \\ -R\sin\theta\sin\delta & R\cos\theta\cos\delta & \sin\theta\cos\delta \\ R\cos\delta & 0 & \sin\delta \end{bmatrix} \begin{bmatrix} d\delta \\ d\alpha \\ dR \end{bmatrix}. \qquad (89)$$

Finally, to compute the error ellipsoid for the point Q, we try to find a system of axes such that no correlation exists between them, i.e., so that the variance matrix will be written as

$$\begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix}.$$

Let T be the matrix of the rotation. T is a 3×3 orthonormal matrix, and its elements can be expressed as a function of three quantities (e.g., ϕ, λ and A).

If $V_0\{X_Q\}$ is the variance of X_Q after the rotation T, we will have

$$V_0\{X_Q\}=T\cdot V\{X_Q\}\cdot T'=\begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix} \tag{93}$$

Solving equation (93) we can find both a^i and T; a^i is obtained from the solution of the discriminating cubic:

$$|V\{X_Q\}-a^iI|=0, \tag{94}$$

and T is obtained from the solution of the system:

$$(V\{X_Q\}-a^i)t_i=0$$
$$t_i't_i=1 \tag{95}$$

where t_i are the column vectors of T (or principal directions).

Although we must have the matrix A to be able to find the variance X_Q, every observation will define completely the position of the station if all three elements α', δ', and r have been measured. Or, every observed quantity will define one locus which in the case of small corrections dX^i_Q, will be a plane. To get higher accuracy for X^i_Q, these planes must intersect in right angles if possible.

If we measure only distances r and the given positions S lie on a straight line, there is no solution (the planes will intersect in one line). The closer the positions S are to a straight line, the weaker the solution will be. The solution will be weak also if the given positions and the station lie on the same plane.

Furthermore, the last part of equation (84) (if we assume the positions of the object as given and correct) can be written:

$$\frac{dr}{\bar r}=-\frac{\cos\bar\theta\cos\bar\delta}{\bar r}dX^1-\frac{\sin\bar\theta\cos\bar\delta}{\bar r}dX^2-\frac{\sin\bar\delta}{\bar r}dX^3. \tag{96}$$

Therefore the system (84) can be written as shown in equation (84c), the matrix being orthonormal.

As we said previously, $d\delta'$ and $d\alpha'\cos\delta'$ will in general have the same weight. If dr/r also has the same weight,[4] the variance of X^i_Q as obtained from equation (84c) will be

$$V\{X_Q\}=\sigma^2r^{\cdot 2}I, \tag{97}$$

σ being the standard error of the weight unit.

This proves that the accuracy in the determination of the station Q is independent of the net configuration and the error ellipsoid is a sphere (the error, however, will be proportional to the distance).

The same is true if we have more than one observed position of the object (provided we have observations of the same weight). In this case we see that the matrix A is composed

[4] In most of the cases the errors in distance measurements are supposed to be proportional to the distance. In this case the weight of $dr/\bar r$ will be the same for all distances. In the electronic methods of distance measurements, the errors are sometimes supposed to be independent of the distance. In this case $dr/\bar r$ will not have the same weight for all r's.

$$\begin{bmatrix} d\delta' \\ d\alpha'\cos\delta' \\ \dfrac{dr}{\bar r}\end{bmatrix}=\frac{1}{\bar r}\begin{bmatrix} \cos\bar\theta'\sin\bar\delta' & \sin\bar\theta'\sin\bar\delta' & -\cos\bar\delta' \\ \sin\bar\theta' & -\cos\bar\theta' & 0 \\ -\cos\bar\theta'\cos\bar\delta' & -\sin\bar\theta'\cos\bar\delta' & -\sin\bar\delta' \end{bmatrix}\begin{bmatrix} dX^1_Q \\ dX^2_Q \\ dX^3_Q\end{bmatrix}. \tag{84c}$$

of orthonormal matrices with a factor $1/r^j$, so we can easily prove that

$$A'\cdot A=\left(\frac{1}{r^j}\cdot\frac{1}{r^j}\right)I$$

or

$$[A'\cdot A]^{-1}=\left(\frac{1}{r^j}\cdot\frac{1}{r^j}\right)^{-1}I.$$

Introducing this expression to equation (92a), we obtain

$$V\{X_Q^j\}=\frac{1}{s}\sigma^2\bar{r}^2\cdot I \tag{98}$$

where $\bar{r}^2$ is the harmonic mean of $(r^j)^2$. Equation (98) is of the same type as equation (97) and thus the same conclusions can be obtained.

We can therefore state the following general rules concerning the accuracy of our determinations of position:

a) When we have observed only distances (having the same weight), we obtain the best result if the positions of the observed object can be grouped by threes so that the lines connecting them with the station are mutually perpendicular.

b) When we have observed δ' and α' (and δ' and α' cos δ' have the same weight), we obtain the best result if the positions of the observed object can be grouped by twos, so that the lines connecting them with the station are perpendicular (the distances being assumed to be almost equal).

c) When we have observed all three elements (i.e., r, δ', α'), the net configuration will not affect the accuracy, provided we have the same weights.

d) If we have to choose between measuring distances and directions, we must compare the weights; that is, compare

$$p_1=\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma^2_{\alpha\cos\delta}} \quad \text{with} \quad p_2=\frac{1}{\left(\frac{\sigma_r}{r}\right)^2},$$

where σ refers to standard errors.

If $p_1>p_2$ it will be more accurate to measure direction, and if $p_1<p_2$, it will be more accurate to measure distances.

In most cases $\sigma_\delta=\sigma_{\alpha\cos\delta}=\sigma$. Then $p_1=\frac{2}{\sigma^2}$, and so we compare σ with $\frac{\sigma_r}{r}\sqrt{2}$, and if $\sigma<\frac{\sigma_r}{r}\sqrt{2}$, the measurement of directions is more advantageous, while if $\sigma>\frac{\sigma_r}{r}\sqrt{2}$ the opposite is true.

As an example, if δ' and α' cos δ' can be measured with an accuracy of $\pm 3''$ or about 1.45×10^{-5}, the measurement of distance will be better only if the distances can be measured with an accuracy better than $\frac{\sqrt{1.45}}{2}\,10^{-5}\simeq 10^{-5}$.

Let us now consider the effect of errors in the given points (in this case the object points).

Let δX_S^j be the errors in the position of the object. Then the system (90) will be, with the help of system (84),

$$A_j^q dX_Q^j=l^q+\Phi_j^q\delta X_S^j, \tag{99}$$

where Φ is a $q\times q$ matrix having in its diagonal the matrices $-\Theta$ and the other elements zero. If all the elements have not been observed, the matrices Θ will not be square, and thus the matrix Φ will not be square either.

This means that we introduce the errors δX_Q^j in X_Q^j which arise from the errors δX_S in the given positions X_S:

$$A\cdot\delta X_Q=\Phi\cdot\delta X_S. \tag{100}$$

We want to find the variance $V\{X_Q\}$ of δX_Q (or of X_Q), knowing the variance $V\{X_S\}$ of δX_S (or of X_S). From equation (100) we have

$$V\{\delta X_Q\}=V\{X_Q\}=[A\cdot V^{-1}\{\Phi\cdot\delta X_S\}A]^{-1},$$

or

$$V\{X_Q\}=[A'(\Phi\cdot V\{X_S\}\Phi')^{-1}A]^{-1}. \tag{101}$$

If $V\{X_S\}=\sigma_S^2\cdot I$, equation (101) will be simplified to

$$V\{X_Q\}=\sigma_S^2[A'(\Phi\Phi')^{-1}A]^{-1}. \tag{102}$$

Considering that the matrices A and Φ are made from the submatrices $-\Theta_j$, we have the matrices in (102a). Therefore

$$V\{X_Q\}=\frac{\sigma_S^2}{s}I. \tag{103}$$

Or, the variance of X_Q from the standard error σ_S in the position of the given points is independent of the net configuration, and the standard error of X_Q is equal to the standard error of X_S divided by $\sqrt{s}$, the result to be expected.

Observations to objects of unknown positions

We shall now discuss the computation of position for the case in which we do not know the position of the object, but have observed it from stations of both known and unknown positions. Since the object is moving, the observations must be made simultaneously.

For one solution, we could first compute the positions of the object by using observations from the known stations, and eventually make an adjustment if we have superfluous observations. Then from the already computed positions of the object, we could find (or adjust for) the positions of the unknown stations by using the observations made from the unknown stations to the object.

It will be more advantageous, however, if we adjust the space net as a whole, since we will get higher accuracy.

Unconditioned equations of observation.—Let us suppose we have n known stations Q_k, m unknown stations Q_u and s unknown positions of the object S_j. Let us also suppose that we have observed the object from both known and unknown stations, but have not necessarily observed all its positions nor all of the elements δ', α', r.

For every element observed, we will have an observation equation relating the position of the station and the object that will be of the form of one of equations (84).

We will call Θ_{jk} the matrix of equation (84) connecting the j position of the object with the known station k, and Θ_{ju} the matrix connecting the j position of the object with the unknown station u.

The matrix Θ_{ju} or Θ_{jk} may be the complete matrix of equation (84) or it may be incomplete, that is, it may contain only certain rows of that matrix, depending on how many and what elements have been observed from the station. We will then have a system of observation equations of the form

$$A\cdot dX=l, \tag{104}$$

where A is a matrix consisting of submatrices Θ_{jk} and Θ_{ju}, dX a vector consisting of the vectors $dX^i_{Q_u}$ and $dX^i_{S_j}$ (corrections to the approximate coordinates $\overline{X}^i_{Q_u}$ and $\overline{X}^i_{S_j}$) and l a vector consisting of the vectors $(d\delta', d\alpha', dr)$ obtained from the observations at both known and unknown stations. Needless to say, these vec-

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$$\Phi\Phi'=\begin{bmatrix}-\Theta_1 & 0 & \cdots\\ 0 & -\Theta_2 & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}-\Theta_1' & 0 & \cdots\\ 0 & -\Theta_2' & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}=\begin{bmatrix}\Theta_1\Theta_1' & 0 & \cdots\\ 0 & \Theta_2\Theta_2' & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}$$

or

$$(\Phi\Phi')^{-1}=\begin{bmatrix}(\Theta_1\Theta_1')^{-1} & 0 & \cdots\\ 0 & (\Theta_2\Theta_2')^{-1} & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix} \tag{102a}$$

and

$$[A'(\Phi\Phi')^{-1}A]=(-\Theta_1'-\Theta_2'\ldots)\begin{bmatrix}(\Theta_1\Theta_1')^{-1} & 0 & \cdots\\ 0 & (\Theta_2\Theta_2')^{-1} & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}-\Theta_1\\ -\Theta_2\\ \cdots\end{bmatrix}=sI.$$

tors need not have all three components; they will depend on how many and what elements we have observed.

The matrix A will be of the general form shown in equation (104a).

We notice that in each of the three partitions (the partition with all zeros is excluded), at every row, there is only one submatrix Θ, and in the two upper partitions, for every element $-\Theta_{ju}$ in the left partition, there is a corresponding element Θ_{ju} on the right, in the same row. An example of a matrix A appears in equation (104b).

The solution by the method of least squares for dX will be given by the system of normal equations

$$N\cdot dX=A'Pl, \tag{105}$$

where $N=A'\cdot P\cdot A$, and $P=$ the weight matrix (if the observations are of different weight). From the general form of the matrix A we can see that the matrix N of normal equations will be of the form shown on page 131. Since the subscripts in parentheses are constant numbers, it is understood that there is no summation with respect to those indices. The matrices P are the corresponding weight matrices.

The matrix N (equation 106) is symmetric, as would be expected. To every row and every column there corresponds one point Q_u and S_j in sequence. Then every element of the matrix is formed by summing all the terms of the form $\Theta' P\Theta$ where Θ is the matrix that relates the two points that correspond to the column and row of that element. Since no observations have been made between the stations or between the objects, the corresponding elements will be zero.

It may happen, however, that we also have observations between the unknown stations. In this case the elements of the matrices that correspond to those stations will not be zero. These observations in practice would be mainly distance measurements by Shoran or any other electronic method, since no intervisibility will exist for observations of directions, because of the long distance between the stations. Such distance measurements between the stations by any electronic method will be very desirable because they will strengthen our net.

$$A=\left\{\begin{array}{c|c} \begin{matrix}\text{Submatrix}\\ \text{with elements}\\ -\Theta_{ju}\end{matrix} & \begin{matrix}\text{Submatrix}\\ \text{with elements}\\ +\Theta_{ju}\end{matrix} \\ \hdashline \cdots 0 \cdots & \begin{matrix}\text{Submatrix}\\ \text{with elements}\\ +\Theta_{jk}\end{matrix} \end{array}\right\}\begin{array}{l}\text{Observations from}\\ \text{unknown stations}\\ \\ \text{Observations from}\\ \text{known stations}\end{array} \tag{104a}$$

$$A=\left\{\begin{array}{cccccc} -\Theta_{(j1)(u1)} & \cdot & \cdot & +\Theta_{(j1)(u1)} & \cdot & \cdot \\ -\Theta_{(j2)(u2)} & \cdot & \cdot & \cdot & +\Theta_{(j2)(u2)} & \cdot \\ \cdot & -\Theta_{(j1)(u2)} & \cdot & +\Theta_{(j1)(u2)} & \cdot & \cdot \\ \cdot & \cdot & -\Theta_{(j3)(u3)} & \cdot & \cdot & +\Theta_{(j3)(u3)} \\ \cdot & \cdot & -\Theta_{(j2)(u3)} & \cdot & +\Theta_{(j2)(u3)} & \cdot \\ \hdashline \cdot & \cdot & \cdot & +\Theta_{(j1)(k1)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & +\Theta_{(j1)(k2)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +\Theta_{(j2)(k2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +\Theta_{(j3)(k1)} \end{array}\right\}\begin{array}{l}\text{Observations from}\\ \text{unknown stations}\\ \\ \text{Observations from}\\ \text{known stations}\end{array} \tag{104b}$$

If we observed all three elements from every station, and they have the same weight, the matrix N will be much simplified. The expression $[\Theta'_{ju}P\Theta_{ju}]$ will be $(\nu/\bar{r}^2)I$, where ν is the total number of observed positions and $\bar{r}^2$ is the harmonic mean of the squares of the corresponding distances, the standard error being unity. In this case the matrix N will be as shown in equation (107).

$$N=\left[\begin{array}{ccccccccc}
\Theta'_{j(u1)}P\Theta_{j(u1)} & 0 & \cdots & 0 & -\Theta'_{(f1)(u1)}P\Theta_{(f1)(u1)} & -\Theta'_{(f2)(u1)}P\Theta_{(f2)(u1)} & \cdots & -\Theta'_{(fs)(u1)}P\Theta_{(fs)(u1)} \\
0 & \Theta'_{j(u2)}P\Theta_{j(u2)} & \cdots & 0 & -\Theta'_{(f1)(u2)}P\Theta_{(f1)(u2)} & -\Theta'_{(f2)(u2)}P\Theta_{(f2)(u2)} & \cdots & -\Theta'_{(fs)(u2)}P\Theta_{(fs)(u2)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \Theta'_{j(um)}P\Theta_{j(um)} & -\Theta'_{(f1)(um)}P\Theta_{(f1)(um)} & -\Theta'_{(f2)(um)}P\Theta_{(f2)(um)} & \cdots & -\Theta'_{(fs)(um)}P\Theta_{(fs)(um)} \\
-\Theta'_{(f1)(u1)}P\Theta_{(f1)(u1)} & -\Theta'_{(f1)(u2)}P\Theta_{(f1)(u2)} & \cdots & -\Theta'_{(f1)(um)}P\Theta_{(f1)(um)} & \Theta'_{(f1)u}P\Theta_{(f1)u}+\Theta'_{(f1)k}P\Theta_{(f1)k} & 0 & \cdots & 0 \\
-\Theta'_{(f2)(u1)}P\Theta_{(f2)(u1)} & -\Theta'_{(f2)(u2)}P\Theta_{(f2)(u2)} & \cdots & -\Theta'_{(f2)(um)}P\Theta_{(f2)(um)} & 0 & \Theta'_{(f2)u}P\Theta_{(f2)u}+\Theta'_{(f2)k}P\Theta_{(f2)k} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\Theta'_{(fs)(u1)}P\Theta_{(fs)(u1)} & -\Theta'_{(fs)(u2)}P\Theta_{(fs)(u2)} & \cdots & -\Theta'_{(fs)(um)}P\Theta_{(fs)(um)} & 0 & 0 & \cdots & \Theta'_{(fs)u}P\Theta_{(fs)u}+\Theta'_{(fs)k}P\Theta_{(fs)k}
\end{array}\right] \quad (106)$$

$$N=\left[\begin{array}{cccccccc}
\left[\left(\frac{\nu}{\bar{r}^2}\right)_{u1}I\right] & 0 & \cdots & 0 & \left[-\frac{1}{r_{11}^2}I\right] & \left[-\frac{1}{r_{21}^2}I\right] & \cdots & \left[-\frac{1}{r_{s1}^2}I\right] \\
0 & \left[\left(\frac{\nu}{\bar{r}^2}\right)_{u2}I\right] & \cdots & 0 & \left[-\frac{1}{r_{12}^2}I\right] & \left[-\frac{1}{r_{22}^2}I\right] & \cdots & \left[-\frac{1}{r_{s2}^2}I\right] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \left[\left(\frac{\nu}{\bar{r}^2}\right)_{um}I\right] & \left[-\frac{1}{r_{1m}^2}I\right] & \left[-\frac{1}{r_{2m}^2}I\right] & \cdots & \left[-\frac{1}{r_{sm}^2}I\right] \\
\left[-\frac{1}{r_{11}^2}I\right] & \left[-\frac{1}{r_{12}^2}I\right] & \cdots & \left[-\frac{1}{r_{1m}^2}I\right] & \left[\left(\frac{\nu}{\bar{r}^2}\right)_{f1}I\right] & 0 & \cdots & 0 \\
\left[-\frac{1}{r_{21}^2}I\right] & \left[-\frac{1}{r_{22}^2}I\right] & \cdots & \left[-\frac{1}{r_{2m}^2}I\right] & 0 & \left[\left(\frac{\nu}{\bar{r}^2}\right)_{f2}I\right] & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\left[-\frac{1}{r_{s1}^2}I\right] & \left[-\frac{1}{r_{s2}^2}I\right] & \cdots & \left[-\frac{1}{r_{sm}^2}I\right] & 0 & 0 & \cdots & \left[\left(\frac{\nu}{\bar{r}^2}\right)_{fs}I\right]
\end{array}\right] \quad (107)$$

The variance $V\{dX\}$ of the unknowns will be given by the formula: $N^{-1}=(A'\cdot P\cdot A)^{-1}$. In practice, the weights are given in terms of an appropriate weight unit, in order to have simple numbers. Therefore we have

$$V\{dX\}=\sigma^2 N^{-1}, \qquad (108)$$

σ^2 being the variance of the weight unit.

In all our derivations and formulas we have used the terrestrial coordinate system **X**. The same formulas hold for any other system provided we change the matrices Θ as indicated on page 124 ff.

In case we have geodetic rectangular coordinates x^i, equations (104) to (108) are exactly the same, since the matrices Θ will be the same.

The only restriction is that the coordinates x^i of the known stations must be referred to the same system and of course the positions of the unknown stations and of the object will refer to the same system.

Also it must be mentioned again that the values of α' and δ' will be obtained from equations (32), (34), or (35), depending on the system of coordinates used; e.g., if we use the **X** system we will first find the approximate direction cosines $\bar{n}^i$ with the approximate coordinates $\bar{X}^i$ and then with the help of equation (34), we find $\bar{\theta}$ and $\bar{\delta}$.

If we use the **x** system we have to go through the direction cosines q^i of equations (35) and (36), provided we know the values of dA, $d\xi$, $d\eta$. Since in most cases those values will be unknown, the directions in the **x** system will be the same as in the **X**.

We could simplify the computations if all observations were made within a short interval (some days), during which the instantaneous pole will be practically fixed. In this case we could make all the computations in the **Y** system and afterward make the transformation of the adjusted coordinates.

Introduction of conditions.—There may be certain conditions that must be fulfilled between the unknown stations—for example, the distance between the unknown stations must be kept fixed. In making our adjustment we must then impose a number of condition equations between the values dX^i.

Let the equation

$$C\cdot dX=k \qquad (109)$$

be the condition equations in matrix notation, while

$$A\cdot dX=l \qquad (110)$$

is, as previously, the observation equation.

The solution by least squares will be given by the system

$$\begin{pmatrix} N & C' \\ C & O \end{pmatrix}\begin{pmatrix} dX \\ \lambda \end{pmatrix}=\begin{pmatrix} A'Pl \\ k \end{pmatrix}, \qquad (111)$$

where $N=A'\cdot P\cdot A$, the same as in equation (105) and λ are the auxiliary unknowns (correlates) equal in number to the conditions.

From the solution of equation (111) we get the unknowns dX, while their variance is given by the formula,

$$V\{dX\}=\sigma^2[N^{-1}-N^{-1}\cdot C'(C\cdot N^{-1}\cdot C')^{-1}C\cdot N^{-1}], \qquad (112)$$

σ being the standard error of the weight unit.

In most cases, the conditions will relate the coordinates of the unknown stations Q_u. It is more probable that there are fixed lengths or directions between the unknown stations. There may also, however, be conditions between the coordinates of the object (e.g., we may know the distance between two positions of an object if we know its velocity and the elapsed time).

The conditions will be mainly of two kinds: imposed length and imposed direction.

Let us first consider the conditions of imposed length. Let L_0 be the known and fixed distance between any two points Q_1 and Q_2. Let $\bar{L}$ be the computed distance between the approximate positions $\bar{Q}_1$ and $\bar{Q}_2$. We have

$$(L)^2=(\Delta X^1)^2+(\Delta X^2)^2+(\Delta X^3)^2, \qquad (113)$$

where $\Delta X^i=X^i_2-X^i_1$.

Differentiating, we get

$$L\,dL=\Delta X^1(dX^1_2-dX^1_1)+\Delta X^2(dX^2_2-dX^2_1)+\Delta X^3(dX^3_2-dX^3_1)$$

or

$$dL=\frac{\Delta X^1}{L}(dX^1_2-dX^1_1)+\frac{\Delta X^2}{L}(dX^2_2-dX^2_1)+\frac{\Delta X^3}{L}(dX^3_2-dX^3_1),$$

or, further, by using n^i, the direction cosines of the line $\overline{Q}_1\overline{Q}_2$, we obtain

$$dL=n^1(dX^1_2-dX^1_1)+n^2(dX^2_2-dX^2_1)+n^3(dX^3_2-dX^3_1).$$

If $dL=L_0-\overline{L}$, dX^i is the correction to the approximate coordinates $\overline{X}^i$, and so the condition equation is written

$$n^1dX^1_2-n^1dX^1_1+n^2dX^2_2-n^2dX^2_1+n^3dX^3_2-n^3dX^3_1=L_0-\overline{L}. \tag{114}$$

For every imposed fixed length, we have one condition equation of the form (114) which could also be written as

$$n^idX^i_2-n^idX^i_1=L_0-\overline{L}. \tag{114a}$$

If we take the approximate positions $\overline{Q}$ so that $L_0=\overline{L}$ we have

$$n^idX^i_2-n^idX^i_1=0. \tag{115}$$

By comparing equation (114) with the last of equation (84), we see that they are the same. We can write the conditions also in the form shown in equation (116).

We must be sure that we use the direct distance between the two points (also called chord distance), and not the distance along the terrestrial ellipsoid (length of the geodesic).

Next let us consider conditions imposed on the directions. A line connecting two unknown stations may have a known direction that has to be kept fixed in the adjustment. One direction in space has two freedoms, and so for every imposed direction we will have two condition equations.

Instead of using the direction cosines of the imposed direction, we will find it more convenient to use their equivalent elements, i.e., the angles θ' and δ'. Let θ'_0 and δ'_0 be the imposed values and $\overline{\theta}'$ and $\overline{\delta}'$ the computed values from the approximate coordinates. It will not be necessary to derive the condition equations, since we know beforehand that they will be the same as equations (84a) and (84b). We can therefore write the condition equations by using L for the distance between the two points Q_1 and Q_2, as shown in equations (117) and (118).

$$-\left(\frac{\cos\overline{\theta}'\sin\overline{\delta}'}{L}\right)(dX^1_2-dX^1_1)-\left(\frac{\sin\overline{\theta}'\sin\overline{\delta}'}{L}\right)(dX^2_2-dX^2_1)+\left(\frac{\cos\overline{\delta}'}{L}\right)(dX^3_2-dX^3_1)=\delta'_0-\overline{\delta}', \tag{117}$$

$$-\left(\frac{\sin\overline{\theta}'}{L\cos\overline{\delta}'}\right)(dX^1_2-dX^1_1)+\left(\frac{\cos\overline{\theta}'}{L\cos\overline{\delta}'}\right)(dX^2_2-dX^2_1)=\theta'_0-\overline{\theta}'. \tag{118}$$

If the imposed directions of θ'_0 and δ'_0 have been obtained with the help of astronomic observations, as they will be in general, they refer to the Y system. If we are using a different system, the computed $\overline{\theta}'$ and $\overline{\delta}'$ must be obtained by using the Y^i coordinates of the approximate points.

Regarding the condition equation, we must make the following remark: if we have measured the distance, e.g., by Shoran, between two of the unknown stations, we must not use this measurement as a condition equation, but rather as an observation equation. We can correctly use it as a condition equation only if we want the distance to be kept fixed after the adjustment or, perhaps, if the accuracy of the measurement between the stations is much higher than that of the measurement to the object.

The connection of geodetic systems.—One important problem that often arises in geodesy is that of connecting two geodetic systems; that is, to find the relative positions of the two computation ellipsoids or, equivalently, to find the relative positions of the two X systems of coordinates. The connection can be made by com-

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$$(\cos\overline{\theta}'\cos\overline{\delta}')(dX^1_2-dX^1_1)+(\sin\overline{\theta}'\cos\overline{\delta}')(dX^2_2-dX^2_1)+(\sin\overline{\delta}')(dX^3_2-dX^3_1)=L_0-\overline{L} \tag{116}$$

paring the coordinates of common points of the two systems.

If we have observed the positions of an object from stations belonging to both systems, we will be able to make the connection. For that we use the points belonging to the one system as known points, and we find the coordinates of the points belonging to the other system (as if they were unknown), but we introduce the necessary condition equations so that the net will not change as a whole. We then compare the coordinates computed in this way with the coordinates of the same points as given in the second system.

Instead of comparing the coordinates of points of the one system, as computed from the other system, with those originally given, we could compare the coordinates of the object as computed from the two systems. This will be much more simple, since we will have to compute positions of an object from known stations and will have only groups of normal equations with three unknowns.

The relation between the two systems will be given by one translation (3 unknowns), one rotation (3 unknowns), and perhaps one scale factor (1 unknown).

If the two geodetic systems have no orientation errors dA, $d\xi$ and $d\eta$, they will be parallel, since both will be parallel to the terrestrial system (see p. 109). Connecting the two systems will then require only translation, and no rotation. Further, if the two surveys are scaled correctly (or are scaled far better than we can detect), we have only three unknowns. Then only one position of an object, observed and computed from both systems, will give us the solution. More positions will help in eliminating the accidental errors since we can make an adjustment.

From equation (8) we have the expression $x_1^i = X^i - X_{C_1}^i$ for the coordinates of one point in the first system, and $x_2^i = X^i - X_{C_2}^i$ for the coordinates of the same point in the second system; therefore

$$x_1^i - x_2^i = X_{C_2}^i - X_{C_1}^i = X_{C_{2-1}}^i. \qquad (119)$$

Thus the relative position $X_{C_{2-1}}^i$ (which is the same as the relative position of the center of the computation ellipsoid of the second system with respect to the first), is simply the difference of coordinates of the same point computed in the two systems. If we have more points we will take the mean.

If we want to express this translation in terms of the deflections ξ, η, ζ, we will use equation (7). The ξ and η so obtained will be expressed in length units, but we can express them in angular units by using the radii of curvature. We must remember, however, that those deflections are not absolute but relative, i.e., with respect to the computation ellipsoid of the first datum.

If in equation (7) φ_0, λ_0 are the coordinates of the origin of the second datum, the deflections will correspond to that point. If we want the deflections at any other point (φ, λ), we have to replace φ_0 with φ and λ_0 with λ.

The residuals of $X_{C_{2-1}}^i$, obtained from the different positions of the object, should be within the accuracy of the computed positions. If the residuals are bigger, this indicates that the systems also have orientation errors dA, $d\xi$, and $d\eta$, and/or are at different scales.

It is not possible to find the absolute errors in orientation of the two systems, but we can find the relative orientation error of, e.g., the second system with respect to the first.

If we take the first system as reference, we can write equation (119) as follows:

$$x_1^i = x_2^i + X_{C_{2-1}}^i + Gg_{2-1}^i + \epsilon_{2-1}(\Delta x^i), \qquad (120)$$

where $X_{C_{2-1}}^i$ are the coordinates of the origin of the second system with respect to the first, g_{2-1} the orientation errors at the origin of the second system with respect to the first system (computation ellipsoid), and ϵ_{2-1} is the scale difference of the second system with respect to the first.

In equation (120) we have seven unknowns,—three for the translation, three for the rotation, and one for the scale. In most cases the scaling of the two systems will be so much more correct than can be obtained with this method that we may disregard the last term of equation (120).

Every observed and computable position of the object will provide the three equations of (120).

If we disregard the scale difference, two positions of the object can give the solution for the six unknowns. If we have more positions, we can make an adjustment by the method of least squares, and eventually introduce the scale difference. After the adjustment, we could express $X^i_{C_{2-1}}$ in terms of the deflections.

If the two geodetic systems are connected by Shoran, the connection is not complete because, although we have the horizontal positions referred to the same ellipsoid, the heights refer to the geoid, and it is not possible (except with a gravity method) to connect the geoid of the two areas (cf. Bomford, 1952, p. 367).

If such connections exist, we will introduce them to our solution either as condition equations, if the measured distances should be preserved, or as observation equations if the Shoran distance should also be corrected.

First, we assume no errors in orientation. Let L be the distance between the connected points. If $X^i_{C_{2-1}}=x^i_C$, we have

$$L^i=x^i_2+x^i_C-x^i_1=(x^i_2-x^i_1)+x^i_C, \tag{121}$$

and so the distance

$$(L)^2=[(x^i_2-x^i_1)+x^i_C]'[(x^i_2-x^i_1)+x^i_C]. \tag{122}$$

This equation is not linear with respect to the unknown x^i_C and thus can not be used. We will use, therefore, an expansion into the Taylor series:

$$LdL=(x^i_2-x^i_1)\,dx^i_C+x^i_C dx^i_C$$

or

$$\left(\frac{x^i_2-x^i_1+x^i_C}{L}\right)dx^i_C=dL=L-\overline{L}, \tag{123}$$

or

$$\left(\frac{x^1_2-x^1_1+\overline{x}^1_C}{L}\right)dx^1_C+\left(\frac{x^2_2-x^2_1+\overline{x}^2_C}{L}\right)dx^2_C+$$
$$\left(\frac{x^3_2-x^3_1+\overline{x}^3_C}{L}\right)dx^3_C=L-\overline{L}. \tag{123a}$$

For the value of $\overline{x}^i_C$ we can take the mean value obtained before we introduced the condition equation.[5] L is the observed distance and $\overline{L}$ is the distance computed with the approximate value $\overline{x}^i_C$, and can be obtained from equation (122) by replacing x^i_C with $\overline{x}^i_C$.

[5] The term $x_C{}^i$ can also be neglected completely. Since x^i_C is expected to be smaller than 500 meters, while L is bigger than 200 kilometers, x^i_C/L will be of the order of 10^{-3}; therefore the omitted terms amount to $10^{-3}\,dx^i_C$.

It is very important to introduce the correct weight to our equation, if equation (123) is an observation equation.

In general, where we have also the unknowns dA, $d\xi$, $d\eta$, and ϵ, if the vector $(dA_{2-1}, d\xi_{2-1}, d\eta_{2-1})$ is denoted by $g^i{}_{2-1}$, then, through the use of equation (120), equation (121) will become:

$$L^i=x^i_2+x^i_C+Gg^i+\Delta x^i\epsilon-x^i_1. \tag{124}$$

Therefore,

$$(L)^2=[(x^i_2-x^i_1)'+(x^i_C)'+(g^i_{2-2})'G'+$$
$$\epsilon(\Delta x^i)'][(x^i_2-x^i_1)+x^i_C+Gg^i_{2-1}+\Delta x^i\epsilon]. \tag{125}$$

Differentiating and eliminating the second-order terms, we obtain

$$LdL=[(x^i_2-x^i_1)+x^i_C]'dx^i_C+$$
$$[(x^i_2-x^i_1)'G]dg^i_{2-1}+[(x^i_2-x^i_1)'\Delta x^i]\epsilon,$$

or:

$$\frac{1}{L}[x^i_2-x^i_1+x^i_C]'dx^i_C+\frac{1}{L}[(x^i_2-x^i_1)'G]dg^i_{2-1}+$$
$$\frac{1}{L}[(x^i_2-x^i_1)'\Delta x^i]\epsilon=L-\overline{L}. \tag{126}$$

Equation (126) is the observation equation (or condition equation), the same as (123) but it contains all seven unknowns. We will have as many equations of this form as there are distances measured between the two systems to be connected.

Observations to orbiting objects

If the observed object is orbiting and the orbit is assumed to be known, we could treat the problem as one involving an object of known position (p. 122 ff.). If the orbit is completely unknown (as for a rocket's orbit in the earth's atmosphere), we could treat the problem as one involving an object of unknown position (p. 129 ff.), by making simultaneous observations. In the latter case, however, we would make continuous observations of the object and, having also the time of the observations, we can interpolate for fictitious simultaneous observations. We will call this method that of quasi-simultaneous observations.

But very frequently we will meet the problem in which we know the theory of the orbit (more or less), but we do not know the values of the parameters that define it. It is then possible, from observations from known stations, to find those parameters and eventually to find the positions of unknown stations. Since the object that we can use in this way will be an artificial satellite, we will limit ourselves to elliptic (or quasi-elliptic) orbits.

Quasi-simultaneous observations.—We assume that we have continuous observations of the object from both known and unknown stations and for the same period of time. The result of those observations can be either a continuous recording against time or individual values at, preferably, equal and short intervals of time.

If we have a recording, we can easily find the value that corresponds to a given time T_i. We will then take the times T_1, T_2 . . . and find the values that would have been observed from the different stations at those times. We then apply the method devised for simultaneous observations.[6]

If the data are given as individual observations, we must make a numerical interpolation; we may use a linear interpolation or second and third differences, our choice depending on how smooth the values run. Any interpolation formula can be used, although if the observations are not at equal intervals, we must use Newton's or Lagrange's formulas for unequal intervals.

We could also extrapolate the orbit beyond the observed values. This is dangerous, however, unless we are very sure that the values will continue to run smoothly. Even then, the extrapolation should not be made for an argument far outside the two extreme observed values.

The accuracy that we can get depends on the accuracy of the timing, as well as on the factors mentioned earlier (p. 129 ff.). If we want the position of the object to be accurate within $\pm\ \delta X$ and the velocity of the object is V, the timing accuracy of the observations at all stations should be better than within δT, where $\delta T \leq \frac{\delta X}{V}$.

The interpolation itself will not introduce errors, provided that the observed values run smoothly, that there is no maximum or minimum or inflection point, and that the interpolated interval is small. On the contrary, since every fictitious observation will be obtained from a number of true observations, this method will tend to reduce the effect of accidental errors. On the other hand, we must consider the correlation between the observed values (see p. 115).

[6] It should be pointed out that since the object is moving, to be really simultaneous the observations should refer to the time that the signal (light pulse for optical observations) left the object, and not to the time that the signal was received by the observer.

Use of a first-approximation orbit.—Let us assume the object has an elliptical orbit around the earth, and conforms to Kepler's laws. Unquestionably the orbit of an object not too far from the earth's surface will be much more complicated, but if mean elements are used, the elliptical orbit is very nearly correct for a small interval of time, and can be used as a first approximation.

We will use the following elements to define the orbit at epoch T_0:

mean anomalistic motion	n
eccentricity	e
R.A. of ascending node	Ω
inclination	i
argument of perigee	ω
mean anomaly	M_0

We will use b^u to designate the vector consisting of these elements, in the same sequence; i.e., $b^u = (n, e, \Omega, i, \omega, T_0)$, $(u = 1, 2, \ldots 6)$.

Instead of the mean motion, we could use the period P or the semimajor axis a, as we have the relations,

$$n = \frac{2\pi}{P}, \tag{127}$$

$$n^2 a^3 = k_e^2 \mu. \tag{128}$$

Here k_e^2 is the geocentric gravitational constant with the value (Herrick, Baker, and Hilton, 1958):

$$k_e^2 = 1.4350087 \text{ Mm}^3/\text{min}^2, \tag{129}$$

$$\mu = m_\oplus + m_s + \Delta m,$$

where $m_\oplus$ = mass of earth, m_s = mass of orbiting object, and Δm = an augmentation of the masses for the effect of the perturbations.

For an object of negligible mass orbiting around the earth with an orbit of small eccentricity, we have with sufficient approximation the relation (see also equation (129b), p. 144):

$$\mu=1-J\left(\frac{a_{\oplus}}{a}\right)^2\left(1-\frac{3}{2}\sin^2 i\right), \qquad (129a)$$

where $a_{\oplus}$=the earth's equatorial radius, J= the coefficient of second harmonic of the earth's ellipsoid with the value $J=1.624\times10^{-3}$ (Jacchia, 1958a).

When these six elements are given, the orbit is defined and we can find the position of the object at any time T. The positions can be computed in either rectangular or polar coordinates. We shall use the polar coordinates α, δ, and R as more convenient. (Note that the values α and δ are not the apparent but the mean coordinates, referring to the equinox and the obliquity of the epoch for which the elements of the orbit are given.) They are expressed as follows:

$$\alpha=\Omega+\text{arc tan}\,[\cos i \tan(\omega+v)], \qquad (130)$$

$$\sin\delta=\sin i \sin(\omega+v), \qquad (131)$$

$$R=a(1-e\cos E). \qquad (132)$$

Here the auxiliary elements v=true anomaly, E=eccentric anomaly, M=mean anomaly, are given by the formulae

$$\tan\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \qquad (133)$$

$$M=E-e\sin E, \qquad (134)$$

$$M=M_0+n(T-T_0), \qquad (135)$$

and a is given by equation (128).[7]

The classical methods of Laplace and Gauss may be used for determining the elements of the orbit from three observations (Moulton, 1958), although the orbits of the earth's nearby satellites can be determined also with other more simple and less accurate methods. In all those methods, however, we must assume that we know the positions of the stations with respect to the center of gravity of the earth, since the equations of the orbit are given with respect to that point.

[7] The element M_0 can be replaced by T_0, which is both the epoch and the time of a certain perigee crossing. The expressions for this latter case are given in Veis (1958).

By differentiating equations (90) and (92) to (97), and using $l=\omega+v$, we obtain the following:

$$\frac{\partial\alpha}{\partial\Omega}=1$$

$$\frac{\partial\alpha}{\partial i}=-\frac{\tan l\sin i}{1+\tan^2 l\cos^2 i}$$

$$\frac{\partial\alpha}{\partial\omega}=\frac{\cos i}{\cos^2 l(1+\tan^2 l\cos^2 i)}$$

$$\frac{\partial\alpha}{\partial v}=\frac{\cos i}{\cos^2 l(1+\tan^2 l\cos^2 i)}$$

$$\frac{\partial R}{\partial a}=1-e\cos E$$

$$\frac{\partial R}{\partial e}=-a\cos E$$

$$\frac{\partial R}{\partial E}=a\,e\sin E$$

$$\frac{\partial M}{\partial n}=(T-T_0)$$

$$\frac{\partial M}{\partial T_0}=-n$$

$$\frac{\partial M}{\partial M_0}=1$$

$$\frac{\partial\delta}{\partial i}=\frac{\cos i\sin l}{\cos\delta}$$

$$\frac{\partial\delta}{\partial\omega}=\frac{\sin i\cos l}{\cos\delta}$$

$$\frac{\partial\delta}{\partial v}=\frac{\sin i\cos l}{\cos\delta}$$

$$\frac{\partial v}{\partial E}=\frac{\sin v}{\sin E}$$

$$\frac{\partial v}{\partial e}=\frac{\sin v}{1-e^2}$$

$$\frac{\partial E}{\partial M}=\frac{1}{1-e\cos E}$$

$$\frac{\partial E}{\partial e}=\frac{\sin E}{1-e\cos E}$$

$$\frac{\partial a}{\partial n}=-\frac{2a}{3n}$$

$$\frac{\partial R}{\partial n}=-\frac{2}{3}\frac{a(1-e\cos E)}{n}$$

But

$$d\delta=\left(\frac{\partial\delta}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial n}\right)dn+\left(\frac{\partial\delta}{\partial v}\frac{\partial v}{\partial e}+\frac{\partial\delta}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial e}\right)de+\left(\frac{\partial\delta}{\partial i}\right)di+\left(\frac{\partial\delta}{\partial\omega}\right)d\omega+\left(\frac{\partial\delta}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial M_0}\right)dM_0, \tag{136}$$

$$d\alpha=\left(\frac{\partial\alpha}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial n}\right)dn+\left(\frac{\partial\alpha}{\partial v}\frac{\partial v}{\partial e}+\frac{\partial\alpha}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial e}\right)de+\left(\frac{\partial\alpha}{\partial\Omega}\right)d\Omega+\left(\frac{\partial\alpha}{\partial i}\right)di+\left(\frac{\partial\alpha}{\partial\omega}\right)d\omega+\left(\frac{\partial\alpha}{\partial v}\frac{\partial v}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial M_0}\right)dM_0, \tag{137}$$

$$dR=\left(\frac{\partial R}{\partial n}+\frac{\partial R}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial n}\right)dn+\left(\frac{\partial R}{\partial e}+\frac{\partial R}{\partial E}\frac{\partial E}{\partial e}\right)de+\left(\frac{\partial R}{\partial E}\frac{\partial E}{\partial M}\frac{\partial M}{\partial M_0}\right)dM_0. \tag{138}$$

When we introduce the values of the partial derivatives, we obtain equations (139), (140), and (141).

We now write equations (139) to (141) in a matrix form:

$$\begin{bmatrix} d\delta \\ d\alpha \\ dR \end{bmatrix}=U^k_u\begin{bmatrix} dn \\ de \\ d\Omega \\ di \\ d\omega \\ dM_0 \end{bmatrix}. \tag{142}$$

Or, if we introduce the vector $db^u=(dn,\ de,\ d\Omega,\ di,\ d\omega,\ dM_0)$, we obtain

$$\begin{bmatrix} d\delta \\ d\alpha \\ dR \end{bmatrix}=U^k_u db^u. \tag{143}$$

Equation (143) relates the differentials of the position of the object with the differentials of the parameters of the orbit b^u.

If we introduce equation (143) into equation (84b), we obtain

$$\begin{bmatrix} d\delta' \\ d\alpha' \\ dr \end{bmatrix}=\Theta^r_i\cdot S^i_k\cdot U^k_u\cdot db^u-\Theta^r_i\cdot dX^i_Q. \tag{144}$$

This equation will help us to find the orbit from observations from known stations, with

$$d\delta=\left[\frac{\sin i\cos l\sin v\,(T-T_0)}{\cos\delta\sin E(1-e\cos E)}\right]dn+\left[\frac{\sin i\cos l\sin v}{\cos\delta}\left(\frac{1}{1-e^2}+\frac{1}{1-e\cos E}\right)\right]de+\left[\frac{\cos i\sin l}{\cos\delta}\right]di+\left[\frac{\sin i\cos l}{\cos\delta}\right]d\omega+\left[\frac{\sin i\cos l\sin v}{\cos\delta\sin E\,(1-e\cos E)}\right]dM_0, \tag{139}$$

$$d\alpha=\left[\frac{\cos i\sin v\,(T-T_0)}{\cos^2 l(1+\tan^2 l\cos^2 i)\sin E(1-e\cos E)}\right]dn+\left[\frac{\cos i\sin v}{\cos^2 l(1+\tan^2 l\cos^2 i)}\left(\frac{1}{1-e^2}+\frac{1}{1-e\cos E}\right)\right]de+d\Omega+\left[-\frac{\tan l\sin i}{1+\tan^2 l\cos^2 i}\right]di+\left[\frac{\cos i}{\cos^2 l(1+\tan^2 l\cos^2 i)}\right]d\omega+\left[\frac{\cos i\sin v}{\cos^2 l(1+\tan^2 l\cos^2 i)\sin E\,(1-e\cos E)}\right]dM_0, \tag{140}$$

$$dR=\left[-\frac{2a(1-e\cos E)}{3n}+\frac{ae\sin E}{1-e\cos E}\,(T-T_0)\right]dn+\left[-a\cos E+\frac{ae\sin^2 E}{1-e\cos E}\right]de+\left[\frac{ae\sin E}{1-e\cos E}\right]dM_0. \tag{141}$$

the method of variation of coordinates. In this case $dX_Q^i=0$ and thus

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta \cdot S \cdot U \cdot db^{*}. \qquad (145)$$

From an approximate orbit $\bar{b}^{*}$ we find the elements of the matrices Θ, S, U, as well as the computed quantities $\bar{\delta}$,′ $\bar{\alpha}$,′ r. If we have measured δ', α', r for two positions of the object[8], we can solve for db^{*}. If we have observed only the directions, we need three positions. If we have more observations, we will adjust by the method of least squares.

We can also introduce as unknowns the correction dX_Q^i to the coordinates of the stations and solve simultaneously for both db^{*} and dX_Q^i, using equation (144).

If the stations from which the observations have been made belong to the same geodetic system and there is no orientation error, the value dX_Q^i will be the same for all the stations; therefore we can introduce only one dX^i which will be identical with X_C^i. In this case the observation equations will be

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta \cdot S \cdot U \cdot db^{*} - \Theta \cdot X_C^i, \qquad (146)$$

with a total of 6+3=9 unknowns.

If we introduce also the orientation errors $g^j=(dA,\ d\xi,\ d\eta)$, the observation equations will be:[9]

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta \cdot S \cdot U \cdot db^{*} - \Theta \cdot X_C^i - \Theta \cdot G \cdot g^j. \qquad (147)$$

If we have more than the minimum required number of observations, we will adjust by the method of least squares (as described on p. 122 ff. and p. 129 ff). Whenever this method of adjustment is used, it may be necessary to iterate the solution, since the coefficients of the observation equations depend on the approximate orbit.

[8] α' and δ' must not refer to apparent positions but to those corresponding to the epoch for which we want to compute the elements of the orbit.

[9] Compare with equation (120).

The variance of b^{*} (or db^{*}), X_C^i, and g^j will be

$$V\{db^{*}, X_C^i, g^j\} = [(\Theta \cdot S \cdot U, -\Theta, -\Theta \cdot G)' P (\Theta \cdot S \cdot U, -\Theta, -\Theta \cdot G)]^{-1},$$

where $(\Theta \cdot S \cdot U, -\Theta, -\Theta \cdot G)$ is a matrix consisting of the submatrices $\Theta \cdot S \cdot U$, $-\Theta$ and $-\Theta \cdot G$.

The variance $V\{db^{*}, X_C^i, g^j\}$ can be found only after the observations have been made, since it depends on the configuration of the net. We can, however, notice that the elements of the orbit will be more accurately determined if the positions of the object are not concentrated in only one part of the orbit. For the accuracy of the positions of the stations (or of X_C^i and g^j, which is the same), see page 126 ff.

Use of a second-approximation orbit.—To the elliptic orbit (considered on p. 136 ff.) we will introduce the secular perturbations caused by the oblateness of the earth (second harmonic) and by the air drag.

The oblateness of the earth introduces a rotation of the line of nodes on the equatorial plane (regression of Ω) and a rotation of the line of apsides on the plane of the orbit. The two motions will be uniform. They depend both on the orbit (n, e, i) and on the flattening of the earth (or the constant J).

Various formulas have been developed (Brouwer, 1946; Spitzer, 1950; Davis, Whipple, and Zirker, 1956; King-Hele and Gilmore, 1957) for the values of those motions that will be denoted by $\dot{\Omega}$ and $\dot{\omega}$ (the dot denotes derivatives with respect to time). Cunningham (1957) gives the simple formulas

$$\dot{\Omega} = -nJ\left(\frac{a_\oplus}{p}\right)^2 \cos i,$$

$$\dot{\omega} = nJ\left(\frac{a_\oplus}{p}\right)^2 \left(2 - \frac{5}{2}\sin^2 i\right),$$

where $p=a\ (1-e^2)$, the parameter of the ellipse.

The effect of the air drag is to change the shape and dimension of the orbit on its plane, i.e., to change both a (and thus n) and e. Formulas for $\dot{a}$ and $\dot{e}$ are given by Davis, Whipple, and Zirker (1956) for different assumptions. A very interesting result is that the perigee q changes very little.

Assuming a distribution for the densities, we can integrate dq/da numerically and obtain the relation $q=f(a)$ (Jacchia, 1958a).

A secular variation in the inclination will arise from the rotation of the atmosphere, (Sterne, private communication; Merson, King-Hele, and Plimmer, 1959). The inclination will diminish.

The secular variations of the elements can be assumed to be linear over a short period of time, but not in general. The most important effect will be the pseudo-periodic variation in the densities at high altitudes, which has been correlated with variations of the flux of solar radiation (Jacchia, 1959a,b). These variations (at altitudes of 700 km the densities may vary by as much as 100 percent) introduce considerable variation in the acceleration (Veis, 1959).

The existence of a third harmonic term in the earth's potential will also introduce a periodic variation in the orbital elements (O'Keefe and Eckels, 1958).

If we take only the secular part, the orbit could be expressed in the general form of a polynomial,

$$b^*=b_0^*+\dot{b}^*(T-T_0)+\frac{1}{2}\ddot{b}^*(T-T_0)^2+\ldots$$

where $\dot{b}$ is the vector $(\dot{n}, \dot{e}, \dot{\Omega}, \dot{i}, \dot{\omega}, \dot{M}_0)$, $\ddot{b}$ the vector $(\ddot{n}, \ddot{e}, \ddot{\Omega}, \ddot{i}, \ddot{\omega}, \ddot{M}_0)$, etc. The number of terms will depend on the maximum value of $(T-T_0)$. Additional (e.g., exponential or trigonometric) terms can be added if needed.

To find the position of the orbiting object at time T, we will use equations (130) to (134), using for e, Ω, i, ω, the instantaneous values corresponding to time T, and for equation (135) we will integrate

$$M=M_0+\int_{T_0}^{T} n dT.$$

One method that could be applied to determine the orbit (second approximation), similar to the one on page 138 ff., is the following: With the new definition for the orbit, equation (143) will be written as in (143a), where db_0^*, $d\dot{b}^\bullet$, $d\ddot{b}^{**}$ . . . are the corrections to the approximate values $\bar{b}_0^*$, $\bar{\dot{b}}^\bullet$, $\bar{\ddot{b}}^{**}$ . . . (db_0, $d\dot{b}$, $d\ddot{b}$ . . . may not be of the same dimensions; actually $\dot{M}=n$ so that $\dot{b}$ does not contain $\dot{M}$).

Then equation (144) will be written as in (144a).

Assuming the stations known (i.e., $dX_Q^i=0$), we solve for db_0^*, $d\dot{b}^\bullet$ . . . as on page 138. We have applied this method for orbit determinations at the Smithsonian Astrophysical Observatory (Veis, unpublished) and obtained very satisfactory results.

A second method would be to find the mean elements from observations within a short interval of time by using, if necessary, approximate values for the variations of the elements to make the reductions. Provided the time interval is short, errors in the values of $\dot{b}$, $\ddot{b}$ . . . will not have much effect. Given the mean elements for different epochs T_1, T_2, . . ., we can find the value of b as a function of time.

We now consider the simultaneous determination of the orbit and the positions of the stations. If we try to solve equation (144a) for both the orbit and the positions of the stations we will have a large number of un-

$$\begin{pmatrix} d\delta \\ d\alpha \\ dR \end{pmatrix}=[U^k_*, U^k_\bullet(T-T_0), U^k_{**}(T-T_0)^2 \ldots]\begin{pmatrix} db_0^* \\ d\dot{b}^\bullet \\ \tfrac{1}{2}d\ddot{b}^{**} \\ \vdots \end{pmatrix}. \qquad (143a)$$

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix}=\Theta_i^r S_k^i[U^k_*, U^k_\bullet(T-T_0) \ldots]\begin{pmatrix} db_0^* \\ d\dot{b}^\bullet \\ \vdots \end{pmatrix}-\Theta_i^r dX_Q^i. \qquad (144a)$$

knowns, and thus the solution will be weak. Therefore we solve separately for the orbit at epoch and the positions of the stations, and separately for the variation of the elements of the orbit.

The larger the part of the orbit we use and the longer the interval of our observations, the more accurate will be the elements of the orbit, as well as their variations. But on the other hand, the smaller the part of the orbit we use and the shorter the period during which the observations are made, the more likely it is that the computed positions of the orbiting object will be correct. For this reason we will divide the observations into two groups.

In the first group, the observations will be made over a long period of time to determine the variation of the elements $\dot{b}$, $\ddot{b}$, . . . If the observations are made from enough stations (which need not belong to those we use for geodetic purposes; on the contrary, they preferably should be spread all over the world), and for a rather long period of time (many revolutions), they can provide us with accurate enough values for $\dot{b}$, $\ddot{b}$, . . ., although the observations and the positions of the stations may not be so accurate.

The second group of observations will be made as described on page 136 ff. Our purpose will be to find the elements of the mean orbit at epoch and the corrections in the positions of the stations. For this we will use an approximate mean orbit $\bar{b}_0^u$ for an arbitrary epoch T_0 (e.g., the middle of the observations). We will then compute the positions of the orbiting object by using $\bar{b}_0^u+\dot{b}^v(T-T_0)$ as orbit, where $\dot{b}^v$ is the vector obtained from the first group of observations. In most cases, $\dot{b}^v$ will be constant in the interval $(T-T_0)$. If not, we must also include terms of $\ddot{b}^w$, $\dddot{b}^t$, etc.

If we assume there are no errors in the value of $\dot{b}^v$, the discrepancies will be due only to db_0^u, i.e., the correction to the mean orbit of epoch T_0. Thus the observation equations for stations of the same system will be similar to equation (147):

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta\cdot S\cdot U\cdot db_0^u - \Theta\cdot X_C^i - \Theta\cdot G\cdot g^j, \qquad (148)$$

the matrix U being evaluated for the value of

$$b^u = \bar{b}_0^u + \dot{b}^v(T-T_0) + 1/2\,\ddot{b}^w(T-T_0)^2 + \ldots$$

Separating the unknowns as described above may not always be the most efficient method. For each situation we must decide which separation will give the strongest solution. For example, the mean anomaly n will often be more accurately determined from the first group of observations than from the second (it should be remembered that n is actually the secular variation of M).

With the help of equation (146)—where X_C^i will be replaced by dX_O^i, which is the correction to the coordinates of the station—we could use stations not belonging to the same system, but such a method would be less accurate because of the large number of unknowns. In such a case we must also have distance measurements to the object.

From stations of the same geodetic system we will not usually be able to observe the object over a large part of its orbit, and thus the orbit will not be well determined. This inaccuracy will, in turn, bring large errors in the determination of X_C^i and g^j.

There are two possible methods for dealing with the problem.

1) The first method uses two groups of stations, belonging to two different geodetic systems, from which the object can be observed at two different parts of its orbit during the same revolution (fig. 16).

The observation equations will be two groups of the form of equation (148), the one referring

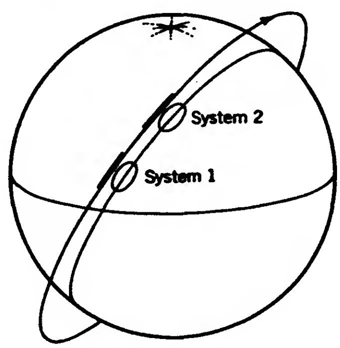

FIGURE 16.—Connection of two geodetic systems by first method.

to observations from stations of the first system and the other to observations from stations of the second system; or,

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta\cdot S\cdot U\cdot db_0^u - \Theta\cdot X_{C_1}^i - \Theta\cdot G\cdot g_1^j,$$

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = \Theta\cdot S\cdot U\cdot db_0^u - \Theta\cdot X_{C_2}^i - \Theta\cdot G\cdot g_2^j. \tag{149}$$

This gives a total of 18 unknowns, or 12 unknowns if the geodetic systems have no orientation errors.

If we have independent unknown stations between the two systems, we could compute their positions by observing the object while it moves from one geodetic system to the other. For this we will compute the position of the object for the times of observation, using the orbit $\bar{b}_0^u + db_0^u + \dot{b}^u\ (T-T_0) + \ldots$, as obtained from the adjustment of equation (149), and continue according to the procedure described on page 122 ff.

Or, we could introduce the observations from the unknown stations to the system (149) and adjust as a whole. This would give a more rigorous solution, but would result in a system with more unknowns. Similarly, if we have more than two geodetic systems along the orbit, we will have to introduce for each one an additional group of equations, of the form (149).

2) The second method is to use a small part of the orbit, which is the same for the observations from both geodetic systems. This can be done since different areas will be under the same part of the orbit during each revolution (fig. 17) because of the rotation of the earth. The orbit determined in this manner may not be very accurate but, on the other hand, the poorly determined elements willl not greatly affect the determination of the positions. The observation equations will be the same as those of equation (149).

Errors in the orbit and the timing will have certain effects. From equations (143) and (89a) we can see that an error δd^u in the orbit will introduce an error in the position of the object,

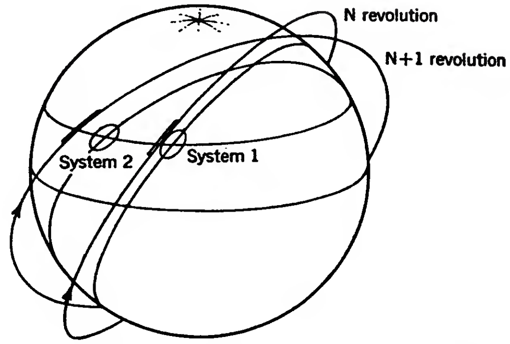

FIGURE 17.—Connection of two geodetic systems by second method.

given by the expression

$$\delta X_S^i = SU\,\delta b^u. \tag{150}$$

To find the effect of an error δT in the timing of the observations, we must differentiate equations (129) to (134) with respect to T so that we can find the effect of dT in α, δ, R. The result will be

$$\begin{pmatrix} d\alpha \\ d\delta \\ dR \end{pmatrix} = u^k dT$$

where u^k is a vector, the last row of the matrix U multiplied by n.

So the introduced errors in the position of the object will be

$$\delta X_S^i = S_k^i u^k \delta T. \tag{151}$$

If, furthermore, we have an error $\delta \dot{b}^u$ in the value of $\dot{b}^u$ (we again use the index u to indicate the complete vector $\dot{b}$, $\ddot{b}$ . . . with zero elements if they do not exist), we will introduce an additional error $\delta \dot{b}^u (T-T_0)$ to $\dot{b}^u$, or an error in the position of the object,

$$\delta X_S^i = (T-T_0) SU \delta \dot{b}^u. \tag{152}$$

From equations (150), (151), and (152) we see that the variance of X_S^i will be

$$V\{X_S^i\} = S\cdot U\cdot V\{b_0^u\}\cdot U'\cdot S' + S\cdot u\cdot V\{T\}\cdot u'\cdot S' + (T-T_0) S\cdot U\cdot V\{\dot{b}^u\}\cdot U'\cdot S'(T-T_0).$$

The variance of X_Q^i, then, can be found with the help of equation (101).

The effect of an error δT in $\dot{b}^u$ $(T-T_0)$ will be of the second order and thus may be neglected.

The accuracy with which we can expect to determine the orbit and the positions of the unknown station (or the elements X_C^i and g^j) depends not only on the accuracy of the observations and the net configuration (see p. 139), but also on the accuracy of the elements $\dot{b}_0^u$ (since they are determined separately) and on the correctness of our theory of the orbit.

An error $\delta\dot{b}_0^u$ in $\dot{b}_0^u$ in the determination of the orbit with the method described on p. 140 ff. will introduce an error to the value of $b_0^u=\bar{b}_0^u+db_0^u$:

$$\delta b_0^u=\delta\dot{b}^u E\{(T-T_0)\}, \tag{153}$$

where $E\{(T-T_0)\}$ is the mean value of $(T-T_0)$ of the different observations used. If we select T_0 as the mean time of the observations, $E\{(T-T_0)\}$ will be zero; thus we will not introduce errors in b_0^u (provided $\ddot{b}^u=0$). However, an error $\delta\dot{b}^u$ will introduce errors δX_S^i in the positions given by equation (152).

We must keep δX_S^i as low as possible, and this can be done by keeping $\delta\dot{b}_0^u$ low (by making many observations from many stations well spread along the orbit and over a long period of time), and by keeping short the interval of time used for the geodetic connections (using no more than a few revolutions, if possible).

If we desire an accuracy such that the errors in position do not exceed μ^i, then $\delta\dot{b}_0^u$ and $T-T_0$ should satisfy the conditions,

$$SU\,\delta\dot{b}_0^u(T-T_0)\leq\mu^i. \tag{154}$$

Also the time must be measured with an accuracy such that from equation (151) we obtain

$$S\,u^k\delta T\leq\mu^i. \tag{155}$$

Finally, some errors result from the fact that our theory for the orbit is not complete. Without a complete theory it is not possible to find these errors. We can, however, consider the following remarks:

1. The second-approximation orbit should not be very far from the true orbit, and if we limit ourselves to the use of a small part, it is more likely to approach the true orbit. Also, the farther from the earth's surface the orbit, the smaller the effect of the perturbations from the earth's gravitational field and from air drag.

2. In our second-approximation orbit we have assumed that the one focus of the instantaneous elliptic orbit is always at the center of gravity of the earth. This assumption may not be correct for the ellipse that we fit to the true orbit. In such a case, the values X_C^i will not be correct, since they will correspond not to the true center of gravity, but to an assumed center of gravity, at the focus of the fitted ellipse. This can be checked by computing X_C^i for the same geodetic system with different orbits. However, although the X_C^i values may not be correct, the relative values between systems connected with observations on the same part of the orbit will be much more nearly correct, since both will refer to the same assumed center of gravity.

3. There are also periodic perturbations that we have neglected, which introduce a periodic variation in the position of the object from the elliptic orbit. The effect of such errors is reduced by the fact that we fit the elliptical orbit to the observed positions. There will be, however, a displacement of the ellipse.

For best results, therefore, we should use only a small part of the orbit. Thus the method given on page 142 is more advantageous. The results will also be improved if we make the same connection, using different orbits.

4. Perturbations caused by the earth's gravitational anomalies are expected to be insignificant for the small interval of time covered by our observations.

5. For the computation of the orbit we have assumed that the constant $k_e^2\mu$ in equation (128) is known and correct. Actually, an uncertainty exists of a little more than 10^{-6} Mm^3/min^2 (Herrick, Baker, and Hilton, 1958); this includes the uncertainty of J. Thus the value of a computed from n may be wrong by about 80 meters, which will introduce an error of the same magnitude in the geocentric positions.

To eliminate this error we can introduce the constant $k_e^2\mu$ as an additional unknown, or, what amounts to the same thing, define the orbit with both a and n. Thus, the orbit will be defined with seven elements; therefore b^u and U_u^k will have different meanings.

The use of a third-approximation orbit.—Until now we have not considered any periodic variations in the orbital elements. However, such periodic variations exist because of the non-sphericity of the earth's gravitational field, and because of the variation of air densities with altitude. These variations may be of long period, ω or 2ω (some months), or of short period, v or $2v$ (of the order of 1 hour).

Pseudo-periodic variations (mainly in n) also exist, due to pseudo-periodic variations in the densities, of a geophysical or solar character.

If we use the methods already described we need not worry about the long period variations, if we limit ourselves (as we do) to observations covering only some days. (The periodic variation of the orbit due to the third harmonic (Kozai, 1959) can easily be taken into account.) The situation is different with the short period variations, however; they can be reduced to some extent only if our observations are very well spread in true anomaly over the orbit.

At the bottom of this page are expressions for the short period variations (or perturbations), reduced to four elements. They were derived by Kozai (unpublished).

Associated with these expressions is the following value for μ:

$$\mu=1-J\left(\frac{a_\oplus}{p}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\,. \tag{129b}$$

These expressions show that the discrepancy from an orbit without short period perturbations can amount to 1 to 2 kilometers, and thus cannot be neglected. Since these variations can be computed from their theoretical values with sufficient accuracy (the effect of an error in J is completely negligible), we will correct the orbital elements for short period perturbations to obtain a third-approximation orbit. Note that it is not an oscillating orbit, and we shall therefore call it a mean orbit. When we use it to compute the position of the object at a certain time, we should add the effect of the short period perturbations to the elements as given from the previous expressions.

Aside from the effect from the nonspherical potential,[10] a short period variation occurs in the orbital elements because the air densities, and thus the air drag, vary with altitude. Indeed, the assumption of a secular variation in n cannot be correct since the greater part of the effect of the air resistance during one revolution will occur within a very small region around perigee. However, the amplitude of this periodic variation cannot be more than the total variation per revolution. In any case, the orbits that will be of any use for geodetic applications will have small values of $\dot{n}$; therefore this periodic effect can be neglected (the displacement along the orbit per revolution will be equal to $\dot{n}/n^2$).

In applying the third-approximation orbit we will use the methods described previously, with one exception. In computing the position of the object at the time of observation, T, in order to determine the computed values $\bar{\delta}'$, $\bar{\alpha}'$, $\bar{r}$, we will not use the approximate orbit $\bar{b}=b_0+\dot{b}(T-T_0)+\ldots$. Instead, we will use the orbit $\bar{b}+\delta b$, where δb are the short period variations of the orbit.

[10] The short period effect of the other terms in the expansion of the gravity potential is very small.

$$\delta i=\frac{1}{4}J\left(\frac{a_\oplus}{p}\right)^2\sin i\left\{\cos 2(v+\omega)+e\cos(v+2\omega)+\frac{1}{3}e\cos(3v+2\omega)\right\}.$$

$$\delta\Omega=-J\left(\frac{a_\oplus}{p}\right)^2\cos i\left\{(v-M)+e\sin v-\frac{1}{2}\sin 2(v+\omega)-\frac{1}{2}e\sin(v+2\omega)-\frac{1}{6}e\sin(3v+2\omega)\right\}.$$

$$\delta R=\frac{1}{3}J\left(\frac{a_\oplus}{p}\right)^2 p\left(1-\frac{3}{2}\sin^2 i\right)\left[-1-\frac{1}{e}\left(1-\sqrt{1-e^2}\right)\cos v+\frac{R}{a}\frac{1}{\sqrt{1-e^2}}\right]+J\left(\frac{a_\oplus}{p}\right)^2 p\sin^2 i\left[\frac{1}{6}\cos 2(v+\omega)\right].$$

$$\delta l=J\left(\frac{a_\oplus}{p}\right)^2\left[\left(2-\frac{5}{2}\sin^2 i\right)(v-M+e\sin v)+\left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{2}{3}e\left(1-\frac{e^2}{2}-\sqrt{1-e^2}\right)\sin v+\right.\right.$$
$$\left.\frac{1}{6}\left(1-\sqrt{1-e^2}\right)\sin 2v\right\}-\left(\frac{1}{2}-\frac{5}{6}\sin^2 i\right)e\sin(v+2\omega)-$$
$$\left.\left(\frac{1}{2}-\frac{7}{12}\sin^2 i\right)\sin 2(v+\omega)-\frac{1}{6}(1-\sin^2 i)e\sin(3v+2\omega)\right].$$

An orbit of unknown mean distance.—We shall now consider the case in which we know the orbit except for the mean parallax (or mean distance, which corresponds to an unknown value for k_e^2). (The moon's orbit is an example; after the perfection of Brown's (1899a, 1899b, 1901, 1908) theory, it is expected that the only doubtful value will be the mean parallax, ϖ_0.)

Since the orbit will be based on the unknown mean distance, there will be an unknown scale factor $(1+\kappa)$ by which the given distances R_t should be multiplied to give the correct distance R. Thus

$$R=(1+\kappa)R_t$$

or

$$dR=R-R_t=\kappa R_t,$$

where κ is supposed to be a small quantity.

But a difference dR in R will introduce a difference in X_S^i given by equation (89); or

$$\begin{aligned} dX_S^1&=\cos\theta\cos\delta\, dR=(R_t\cos\theta\cos\delta)\kappa=s^1\kappa,\\ dX_S^2&=\sin\theta\cos\delta\, dR=(R_t\sin\theta\cos\delta)\kappa=s^2\kappa,\\ dX_S^3&=\sin\delta\, dR \qquad =(R_t\sin\delta)\kappa \qquad =s^3\kappa. \end{aligned} \tag{156}$$

Introducing equation (156) to equation (84) we get

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix}=\Theta s^i\kappa-\Theta dX_Q^i.$$

These are the observation equations for the computation of both dX_Q^i and κ. They are the same as equation (84b) with $d\delta=d\alpha=0$, $dR=R_t\kappa$.

Applications of the various methods

Any of the three methods described earlier can be used to obtain geodetic information from the artificial satellites. The choice of method will depend on the kind of satellite, its orbit, shape, instrumentation, etc.

If we know the orbit of the satellite with a good degree of accuracy (of the same order as that with which we want to determine the positions of the stations), we can apply the method described in pages 122 to 126. If we do not know the orbit accurately enough, but we do know that the variation of the elements is more or less smooth, we can use the method of orbital interpolation as described in pages 135 to 144. This more flexible method can be used with any spherical satellite of rather high specific gravity and a perigee height of more than 500 km.

If we do not know the orbit at all, we can use the satellite by making simultaneous observations and using the method described in pages 129 to 133. Simultaneity of observations can be ideally achieved if the satellite is specially instrumented to send flashes of very short duration; such an object we shall call a flashing satellite.

The orbit: ephemerides and visibility.—Random variations occur in the orbital acceleration,[11] because of variations (of geophysical and solar origin) in the densities. Such random variations make it almost impossible to determine an orbit that could be extrapolated for a long period of time and give a sufficiently accurate position for the satellite. The acceleration varies from 10 per cent (perigee heights of 170 km) to 100 per cent (perigee heights of 700 km) of its value. These variations are sufficient to displace a satellite by several kilometers along the orbit (or some seconds in time) in one day, even for a perigee height of 700 km. However, such variations in acceleration do not seem to be of a sudden character.

For these reasons, the orbits and ephemerides prepared by the various agencies (e.g., the Smithsonian Astrophysical Observatory) are revised at least once every week. Although the extrapolation of an orbit may be in error by several kilometers, the *a posteriori* orbit determination is expected to be much more exact. Preliminary results of orbit determinations made with the method described on page 140 without correction for the short period perturbations and with observations of moderate accuracy, gave residuals in the position of the satellite of the order of 1 km. The final orbits from accurate observations are expected to be accurate to within ± 100 meters, or better.

Visibility is an important factor. Unless we measure only distances, or use electronic meth-

[11] By orbital acceleration we mean the variation of the period P or of the mean motion n, per revolution or per day. In the notation of p. 140 the acceleration will be $\dot{n}$.

ods for determining the directions, the satellites must be visible to be observed or photographed.

During the day a satellite is rarely visible. At night it will be visible if it is illuminated, as would be possible if the satellite carried a light source, either continuous or flashing. O'Keefe (1956) has suggested the use of retrodirective reflectors illuminated from the ground by searchlights.

The simplest solution would be to let the sun illuminate the satellite. Indeed the satellite can be out of the earth's shadow and thus illuminated by the sun, while an observer underneath it is in the shadow (fig. 18). The

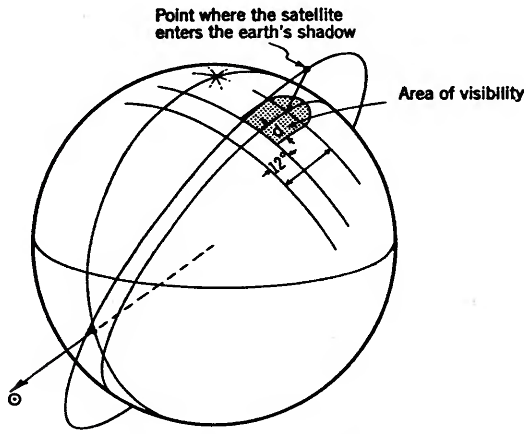

FIGURE 18.—Area of satellite visibility.

satellite will enter into the shadow of the earth when at an angular distance D from the line of the terminator given by the formula

$$\cos D = \frac{\rho}{\rho + H},$$

where ρ is the radius of the earth. The effect of refraction has been excluded.

The observer, on the other hand, should be in darkness, the degree depending on the apparent magnitude of the satellite. If the satellite is rather bright, the nautical twilight (12° depression of the sun) must have ceased for the observer, but if the satellite is faint, the astronomical twilight (18° depression) should be considered. This means that the observer should be farther than 12° (or 18°) from the terminator.

In addition, the satellite must be above the horizon of the observer, preferably more than 15° above, to reduce the errors from refraction. Or, the angular distance d between the satellite and the observer should be

$$\sin (75° - d) = \frac{\rho \sin 105°}{\rho + H}.$$

Figure 18 shows the form of the visibility area on the earth, which can be found easily with a graphical solution if we use a polar stereographic projection (fig. 19), and assume the earth to be a sphere. A polar stereographic map of the world may be combined with this graphical solution to give the visibility areas on the same map. This method is now in use for visibility predictions at the Smithsonian Astrophysical Observatory (Schilling, 1958).

The observations.—Both optical (photographic) and electronic methods can be used for observing the satellites. For the moment, however, only photographic methods seem to be useful for geodetic purposes, since present electronic methods do not yet give the needed accuracy.

Since the apparent angular velocity is fairly important and satellites are not always very bright, the Smithsonian Astrophysical Observatory constructed a special camera, the Baker-Nunn Satellite Tracking camera, designed by J. G. Baker and J. Nunn under the direction of F. L. Whipple (Henize, 1957). The instrument is a Super-Schmidt F/1 camera with

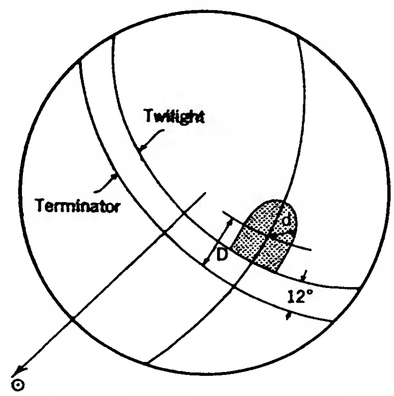

FIGURE 19.—Area of satellite visibility on a stereographic projection.

focal length 50 cm and a field of view $5° \times 30°$, and is expected to photograph satellites fainter than 10th magnitude.

The focal field is spherical, and a cinemascope film stretched on a focal spherical surface is used for the emulsion support. At a scale of 406″ per mm, an accuracy of $\pm 2''$ is expected in the determination of the directions (Henize, 1958).

In addition, a time unit, controlled by a crystal clock, gives the time of the middle of the exposure to $0^s.0001$ (Davis, 1958). However, the accuracy of the timing is not expected to be higher than $\pm 0^s.001$. This error in timing will introduce errors in the direction of the motion, in view of the fact that the satellites have rather rapid motion. The variance of δ' and $\alpha' \cos \delta'$ can be computed from equation (42).

The Baker-Nunn camera was developed especially to photograph faint satellites. If they are not particularly faint we can use astrographic cameras or even long focus refractors, provided we use a special shutter to interrupt the trail and get the timing. For a flashing satellite no shutter is needed.

Markowitz (1959b) has developed a dual-rate camera on the same principle as that of his moon camera (Markowitz, 1954). Ballistic cameras (e.g., of the type made by Wild Heerbrugg Ltd., Switzerland) also can be used for bright satellites. These cameras, developed for missile tracking, can give directions with an accuracy between 3″ and 5″ when properly used. For a flashing satellite they can be used with or without a shutter.

The direction of a satellite can also be determined electronically (see p. 112); there is then no problem of visibility but, on the other hand, the satellite must carry a transmitter. Electronic methods do not as yet give sufficient accuracy and can be used only as described on page 140 ff.

No direct distance measurements to the satellites have yet been made, but are expected in the near future. Johns (1958) discusses this possibility and expects an accuracy of ± 30 meters in the measured distances, a rather optimistic estimate for long distances. If we also include the effect of errors in timing (p. 111), the accuracy will be still less.

The observed directions (or distances) must be corrected for aberration (p. 115 ff.) and refraction (p. 117 ff.).

Method for a known orbit.—The method based on the assumption of a known orbit is perhaps the simplest in regard to the geodetic computations, but it is the most difficult to apply because of the difficulty in obtaining an accurate orbit. The theory of this method is discussed on pages 122 to 128.

If we know the orbit, to determine the position of a station we have only to observe the satellite (or satellites) from the unknown station with an appropriate instrument (e.g., photograph the object with a Baker-Nunn or ballistic camera against the star background). We observe the satellite at a minimum of two positions, and record the times of the observations.

The reduction of the observations, after the appropriate corrections have been applied, will furnish the observed elements (say α' and δ') and their variances. We obtain the solution as described on page 122 ff. The corrections to the approximate coordinates will be given by equation (91) and the variance by equation (92).

If we have a number of stations all belonging to the same geodetic system, we introduce as unknowns the coordinates X^i_C of the origin of the system and, perhaps, the rotation g^j and the scale factor ϵ.

The observation equations using equations (20) and (84) in this case will be:

$$\begin{pmatrix} d\delta' \\ d\alpha' \\ dr \end{pmatrix} = -\Theta'_i X^i_c - \Theta'_i G^i_j g^j - \Theta'_i (X^i - X^i_0)\epsilon. \qquad (157)$$

It is important to know the system of reference in which the orbit is defined and the system in which the observations were made, in order to apply the correct expression and to reduce the observed and computed δ', α', r to the same system, as explained on page 124.

Since this method requires that we know the position of the satellite to a high degree of accuracy, we shall use it only for satellites whose orbits can be determined accurately. Since, as we have seen, random variations in the accelerations displace the satellite considerably along the orbit, we must use an orbit derived from a rather large number of observations made around the same time that the observations from the unknown station (or stations) were made. For a spherical satellite of a high specific gravity and a perigee height of more than 700 km, and preferably of 1000 km, and using good observations, we can get an orbit accurate to ± 100 meters or better, with the method described on page 143. The larger part of this uncertainty will be of an accidental character.

This orbit, and thus the unknown stations as well, will refer to a system of reference that may not be geocentric. Indeed, the origin of this system will be a kind of weighted mean of the origins of the different systems to which the stations, used for the determination of the orbit, were referred. Some complications will also arise from the fact that the true orbit does refer to the center of gravity of the earth.

If we use many stations belonging to different geodetic systems all over the world to determine the orbit, we can expect that a large part of the errors in the different systems will be compensated; thus the origin of the mean system in which the orbit will be given will be close enough to the center of our ideal terrestrial ellipsoid. New positions for the stations could be determined with this orbit and an iteration could be performed to improve the positions of the stations and thus also of the orbit.

Another approach is to compute the orbit by using only stations that belong to one and the same system. The positions of the unknown stations thus determined will refer to that same system.

The larger part of the uncertainty in the position of the satellite will be accidental in character. Therefore if we use a certain number of observations to determine the position of the station, and those observations are distributed in every direction, we will considerably improve the final result as given by equation (84). The same is also true for the effect of errors in the observations. Combining equation (98), which will hold for a good distribution even if we do not measure distances, with equation (103), then for the standard error of the position of a station σ_Q we obtain

$$\sigma_Q^2 = \frac{\sigma_S^2 + \sigma^2 \bar{r}^2}{s}. \tag{158}$$

If we adopt the rather conservative values $\sigma_S = \pm 100$ meters, $\sigma = \pm 4'' = \pm 2 \times 10^{-5}$, $\bar{r} = 3$ megameters, and $s = 25$, we get $\sigma_Q = \pm 23$ meters.

The effect of the systematic errors may increase the error to about ± 30 to 40 meters, this value being always rather conservative.

The velocity of the satellite being about 7–8 km/sec, a systematic error of 1 millisecond in the time (which is rather optimistic) will introduce an error of 7–8 meters in the position. Since this error will be in the direction of the satellite's orbit, the final result can be improved by using different passes of the satellite so that the projections of the orbits on the earth will intersect at different angles.

Finally we have errors arising from errors in the theoretical values for the perturbations of short period (p. 143) which will introduce equivalent errors in the positions.

Since the theory and the values of the constants are good to a few parts in a thousand, and since the amplitude of those perturbations is less than a few kilometers, the errors thus introduced will be less than ± 10 meters. The use of different orbits will reduce the errors still further.

Method of orbital interpolation.—With this method we use the observations to the satellite to determine simultaneously the positions of the stations and the elements of the orbit, as described on pages 129 to 133. This method is very similar to that used for a known orbit. In a final analysis, the only difference is whether or not we solve separately for the orbit and for the positions of the stations. The remarks made for the previous method are thus valid also for this one. The choice of method will depend on the magnitude of the variations of the orbital elements.

This method is best fitted for connecting geodetic systems as described on page 140 ff. The satellite then will be observed during one or two passes over the two systems and the solution will be given by equation (149). The variation of the elements will be determined (see p. 140) from observations from other stations over a longer period of time. It may also prove better to use these auxiliary observations to determine the mean motion n (which is actually the variation of M). Since the observations that will be used for the connection cover a rather short period of time, they may introduce an important uncertainty in n. In this case equation (149) will not include dn as unknown and the vector b^u will be $(e, \Omega, i, \omega, M_0)$. If any other element of the orbit is assumed to be known for any reason, we can eliminate it as unknown from equation (149).

To apply this method, the satellite must be visible during a single revolution over both geodetic systems, and for as long an arc as possible. This condition is not unrealizable, and the higher the position of the satellite, the easier it will be. Such a favorable situation will have a duration of several days and will occur every few months. As an example, we give in figure 20 a favorable condition for a connection between the European and the American geodetic systems. The satellite is assumed to have an inclination of 60° and a height of 500 km. The sun has a declination of +10°, and the difference between Ω and $\alpha_{\odot}$ is 6^h. If we use the second method given on page 142, the visibility conditions are much more favorable.

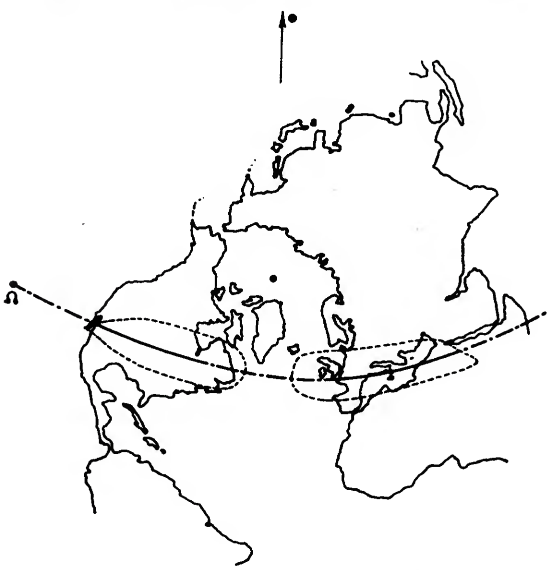

FIGURE 20.—Possible connection of the North American and European geodetic systems. Dash-dot line, satellite orbit; solid line, visible part of orbit; broken line, area of visibility.

The accuracy with which the positions of the stations and the orbital elements will be computed is discussed from a general point of view on pages 139, 140, and 143. For best results, it is essential that we have a good net configuration as described on page 126. If we observe only directions, this will not be very difficult to accomplish. If we measure only distances, it will be rather difficult, and it will be impossible for independent stations. At one station, it is impossible to get mutual perpendicular directions to the satellite during the same revolution. A deviation from the ideal conditions for the net configuration will appreciably reduce the accuracy, and the errors may be as much as five times as large.

The kind of orbit is also very important for the accuracy. The smaller the eccentricity, the less is the effect of the position of the perigee. The higher the perigee of the satellite, the more undisturbed the orbit will be. But since the accuracy of the directions is $\pm 2''$ or 10^{-5}, high orbits will bring large linear error, and should be avoided unless we use long focus cameras. A height of 1,000 to 2,000 km may be a good value.

The final accuracy will depend on the accuracy with which the orbit at epoch could be determined, and the accuracy with which the variation of the orbital elements and the theoretical corrections to the orbit (short period terms) will be given.

The value of $\dot{b}$ can be determined from continuous observations over a long period of time. They need not be of very high accuracy, but should be made from a rather large number of stations.

An accuracy of $\pm 7''$/day can be expected for $\dot{\Omega}$, provided the change of Ω is smooth (the accuracy will depend on the inclination). It follows from equation (152) that the linear accuracy of the satellite's position will be ± 10 meters for a duration $(T-T_0)$ of 1 hour and a satellite height of 700 km. This error will be in a direction perpendicular to the polar axis.

If the timing at the auxiliary stations is taken to $\pm 1^s$, the change of the period can be determined to about $\pm 0^s.025$/day. It follows that the error in the position of the satellite will be ± 7 meters for a value of $(T-T_0)$ equal to one hour. This error will be in the direction of the orbit.

With the help of equation (152), we find that the main term in the error of the position, for an error $\delta\dot{\omega}$ in $\dot{\omega}$ and $\delta\dot{e}$ in $\dot{e}$ will be, respectively,

$$ae(T-T_0)\delta\dot{\omega}$$

and

$$a(T-T_0)\delta\dot{e}.$$

It follows that for a duration $(T-T_0)$ of one hour and a height of 700 km, to get an accuracy of 7 meters in the position of the satellite, we must have

$$\delta\dot{\omega} < \pm 50'' \text{ per day},$$

$$\delta\dot{e} < \pm 0.000024 \text{ per day},$$

the first value being for $e=0.1$.

A very small eccentricity will increase the accuracy of the determination of both $\dot{e}$ and $\dot{\omega}$. Finally, the accuracy will increase if the inclination of the orbit is near 63°4 where $\dot{\omega}$ is zero.

The values of $\dot{b}$ (and $\bar{b}$, etc.) can also be computed from their theoretical expressions, provided the constants are known to a good degree of accuracy. Ω and ω are proportional to J. For the moment, J is known to about 1 part in a thousand, and since those variations are about 5° per day they will be given to about ±0°005. The theoretical determination of $\dot{n}$ and $\dot{e}$ will be more difficult, because the uncertainties of the densities are rather important. For the errors arising from possible errors in the short period terms in the perturbations and systematic errors in the timing, we refer to the method used when the orbit is assumed to be known (p. 147).

The accuracy with which the geodetic connections will be made can be computed only after we have made the observations, because of the very strong correlations between the different elements. A rough estimate can be obtained by making some simplifications such as the following.

Let σ be the (linear) standard error of our observations, i.e., the standard error of the determination of the satellite's position with respect to the geodetic system, the latter assumed to be self-consistent.

If we use a part of the orbit of length L_1 to determine the orbit, and we extrapolate it to a distance L, the errors will be $\sigma(L/L_1)$, and since the orbit is determined from two independent parts (the parts observed from the two geodetic systems) with, say, lengths L_1 and L_2, we will have as errors $\sigma(L/\sqrt{L_1^2+L_2^2})$.

If we have ν observations, considering that we have m unknowns, we get as errors

$$\frac{\sigma}{\sqrt{\nu-m}}\cdot\frac{L}{\sqrt{L_1^2+L_2^2}}.$$

The errors in the determination of the elements for the connection of the two systems will be of the same order as those in the determination of positions.

The value of σ will be of the order of ±20 meters.

If the lengths L_1 and L_2 are equal and the distance between the two systems is $2L_1$, considering that $m=12$, we get as errors

$$\frac{\sigma\sqrt{2}}{\sqrt{\nu-12}}=\frac{28}{\sqrt{\nu-12}} \quad \text{in meters.}$$

If we have a total of 30 observations, we will get an accuracy of about ±7 meters.

This estimate will be valid if we assume the orbital theory to be correct. Including the accidental and systematic errors from other sources, the total error will increase, but in all probability it will stay below ±30 meters. By performing the same geodetic connection with different orbits we should be able to improve our results still more.

Methods for simultaneous observations.—This method does not demand any knowledge of the orbit, but requires only that we make simultaneous observations from two systems. Since the problem is purely geometrical there is no reason to restrict the use of the method to satellites alone. Any visible object [12] at a certain height can serve our purpose.

There are no dynamic parameters entering directly or indirectly into the solution. This fact makes the method a very accurate one

[12] Väisälä (1946) and Atkinson (unpublished) have proposed the use of a flashing rocket in a similar way. For further information on the application of the method to rockets, see Veis (1958).

which may prove to give the best results for geodetic connections or for tieing individual stations (mainly islands) into a geodetic system. The only limitation is the height at which the satellite must be if we are to observe it simultaneously from the two geodetic systems. At an altitude of 1000 km we can connect systems that will be 15° apart, by maintaining a strong net configuration and the elevation angles above 25° to 30° (to keep the uncertainties from refraction as small as possible). If we want to connect geodetic systems that are apart by 30° and 50°, as will be the case for intercontinental ties, the satellites should have altitudes of the order of 5000 km to give strong solutions.

The observations may be reduced to simultaneity either by using the method of quasi-simultaneous observations described on page 136, or by synchronizing the shutters of the different cameras so that they open at the same time. In this case, however, the observations will not, strictly speaking, be simultaneous, because they will not refer to the same time that the light pulse left the satellite (aberrational effect). A reduction could easily be made to correct for this error since we will know approximately the distances and the velocities. The correction will be given by equation (52), which in this case will be written:

$$\epsilon_\alpha = \dot{\alpha}' \frac{\Delta r}{c \cos \delta},$$

$$\epsilon_\delta = \dot{\delta}' \frac{\Delta r}{c},$$

where Δr is the difference in the topocentric distances from the two stations to the satellite.

Simultaneous observations are best obtained when the satellite emits flashes. The techniques for a flashing satellite are discussed in greater detail by Whitney and Veis (1958).

The timing of the simultaneous observations will be needed only if we use the star background to determine the direction (the usual procedure). The timing will be used to relate the sidereal and the terrestrial coordinate systems; thus it does not need to be of very great accuracy since an error of 10 milliseconds will introduce an error in the positions of only 5 meters at the equator. The timing can be secured with an accuracy better than ± 10 milliseconds for a flashing satellite, by various methods (Whitney and Veis, 1958). The methods described in pages 129 to 133 can be used to compute the connections of the geodetic systems, or to determine the position of the independent stations.

To achieve a strong solution it is very important to obtain a good net configuration. We should try to arrange the observed positions of the satellite so that the lines from the known stations to the satellite, as well as the lines from the satellite to the unknown stations, intersect as nearly as possible at right angles.

The final accuracy with which the unknown stations or the elements of the geodetic connection (i.e., X^{i}_{C2-1}, g^{j}, ϵ) will be obtained can be determined only after the final solution, since it depends on the net configuration. We can estimate this accuracy, however, by assuming a perfect net configuration; we can then apply equation (158) by replacing σ_s^2 with

$$\sigma_S^2 = \frac{\sigma^2 \tilde{r}_k^2}{n}$$

in accordance with equation (98), where $\tilde{r}_k^2$ is the harmonic mean of the squares of the distances r from the n unknown stations.

Equation (158) then becomes

$$\sigma_Q^2 = \frac{\sigma^2 \tilde{r}_k^2}{sn} + \frac{\sigma^2 \tilde{r}_u^2}{s},$$

where $\tilde{r}_u^2$ is the harmonic mean of the squares of the distances r from the unknown station.

Assuming that $\sigma = \pm 2'' = \pm 10^{-5}$, $\tilde{r}_u = \tilde{r}_k = 3$ megameters, $s=5$, $n=5$, we get

$$\sigma_Q \simeq \pm 15 \text{ meters.}$$

The effect of systematic errors and an imperfect net configuration will increase the errors to about 20 to 30 meters. By repeating the observations we may get a final accuracy of ± 15 meters.

This value corresponds to a relative accuracy of 10^{-5} for a distance of 1,500 km between the stations to be connected. The existing triangulations, which are used as a sort of base line, are expected to be accurate to only 10^{-5} and may further reduce the final accuracy, especially for the connection of distant stations.

The use of the moon

The moon being a natural satellite of the earth, there is no reason that the methods described for artificial satellites could not apply for the moon as well, although it is by no means the best object for observation, and is somewhat farther from the earth than is desirable for our purposes.

The advantage of using the moon is that, for the present, it is the only terrestrial satellite for which we have a good orbital theory. In the future, when satellites will be launched to heights of some earth radii, they may be expected to replace the moon completely, and advantageously, for geodetic purposes.

Only methods that involve direct observations to the moon are discussed here. For methods involving indirect observations (solar eclipses and occultations of stars), see Lambert (1949), Kukkamaki and Hirvonen (1954), Kalaja (1955), O'Keefe and Anderson (1953), and Army Map Service (1954).

The orbit of the moon: libration.—The presently accepted theory for the moon's orbit is that of Brown (1899a, 1899b, 1901, 1908), who formulated special tables (Brown, 1919) for the preparation of an ephemeris of the moon. The National Ephemerides give the position of the moon (computed with Brown's tables) for every 1^h interval, to $0^s\!.01$ in α, and $0''\!.1$ in δ. With the improvement of electronic computers, the U.S. Naval Observatory undertook to prepare improved ephemerides of the moon, in which the positions were given to $0^s\!.001$ in R.A., $0''\!.01$ in declination, and $0''\!.001$ in parallax, and Brown's theory was used directly without the help of his tables.

In the new ephemeris the empirical (periodic) term of Brown's theory has been removed; the mean longitude has been corrected by an amount

$$-8''\!.72-26''\!.74\ T-11''\!.22\ T^2,$$

to bring accordance with Newcomb's tables of the Sun, Mercury, and Venus; and a small correction has been added to the apparent longitudes to correct for aberration.

The *Improved Lunar Ephemeris* for 1952–59 (U.S. Naval Observatory and Greenwich Royal Observatory, 1954) contains the most accurate material yet available for the position of the moon. The value for the mean horizontal parallax, ϖ_0, may be wrong, however, and for this reason it may be better to introduce an unknown scale factor $(1+\kappa)$. Thus the distance to the moon will be

$$R=\frac{(1+\kappa)}{\sin \varpi_0}\,a_{\oplus}.$$

The ephemeris (as well as the theory) gives the position of the moon as a function of a time that must be absolutely uniform (sometimes called Newtonian). Such a time is, unfortunately, unrealizable for the moment (although the so-called atomic clocks may be able to furnish it). We use, instead, "Ephemeris time" (ET) which is very close to uniform and, as a matter of fact, is computed from observations of the moon. The second of ephemeris time is defined by the I.A.U. as 1:31 556 925.975 of the tropical year 1900.

ET is obtained from UT2 as:

$$\mathrm{ET}=\mathrm{UT2}+\Delta T,$$

where

$$\Delta T=+24^s\!.349+72^s\!.318T+29^s\!.950T^2+1.82144\ B.$$

T is in centuries from 1900.0.

The value of B, determined by direct observations of the moon (Jones, 1939), is $B=l_{☾}-C$, where $l_{☾}$=observed longitude of the moon, and (from Brown's tables)

$$C=-10''\!.71\sin\,(140°.0\ T-240°\!.7)+5''\!.22T^2+12''\!.96\ T+4''\!.65.$$

Using the *Improved Lunar Ephemeris* we have:

$$\Delta T=1.82144\ (l_{☾}-l_E)$$

where l_E is the longitude of the moon given by the new ephemeris.

Ephemeris time can also be computed directly with the observed α and δ by inverse interpolation in the ephemeris. The value obtained from R.A. will be more accurate than the one obtained from the declination (the accuracy will depend on the rate of change of those elements).

But to obtain ET we must have the geocentric observed values of α and δ, which means that we have to know the position of the station, to make the appropriate reductions given by equation (85). We can eliminate the effect of the error in the position of the station by making two observations of the moon at positions differing in hour angle by 12^h. In this case, as can be seen from the second of equation (83), the errors made in the two reductions will be equal and of opposite sign. Thus the mean value of ET computed from those two observations will not be affected by errors in the position of the station.

To make observations of the moon at positions differing by 12^h in hour angle, we must observe at both moonrise and moonset from the same station.

The errors arising from the fact that the R.A. of the moon does not change linearly with ET will be smaller than 0^s001.

The orbital theory gives the position of the center of gravity of the moon, but this is an inaccessible point for observations. Instead, a point called "center of figure" is observed. By that we mean the center of a circle that fits best the apparent limb of the moon.

But the moon does not always show the same limb to the observer because of the libration. The problem then is how to bring the centers of the best fitted circles to the same reference datum, and also, since very often we observe several points on the limb, to bring the topography of the moon to the same reference datum.

Provided the topography of the moon is "accidental" the best fitted circles will have the same radius and the same center. But the topography of the moon has also a systematic character. Yakovkin (1954) finds different radii for different values of libration in latitude (libration in radius), and different radii have been found for the east and west limbs.

The foremost completed material is that of Hayn (1914) and Weimer (1952). A new and more rigorous study, expected to be completed in the near future, has been undertaken by the U.S. Naval Observatory under the direction of Watts (Watts and Adams, 1950). Its purpose is to give the heights of different points of the profile of the moon expressed in seconds of arc with reference to a unique datum, and for different values of the libration.

The datum must finally be connected to the center of gravity of the moon. This connection has not been yet definitely established. However, if the center of figure is used as the datum it probably will not be far from the center of gravity.

The observations.—Markowitz (1954) has devised an ingenious method for measuring the apparent coordinates of the moon, using a special dual-rate moon camera.

The camera, which can be mounted on almost any telescope,[13] takes pictures of the stars for about 20 seconds, the plate following their apparent motion. If at the same time the image of the moon were formed on the photographic plate, the image would not be sharp because of the moon's relative displacement with respect to the stars. To compensate, Markowitz uses a parallel plane plate rotating around an axis that is perpendicular to the direction of the apparent relative orbit, and at an angular velocity such that the image of the moon is kept fixed with respect to the images of the stars. This plate is at the same time a dark filter, which reduces the brightness of the moon.

At the instant at which the parallel plate is perpendicular to the optical axis, and thus the image of the moon is in its true position with respect to the stars, an electric contact affects a chronograph and records the time corresponding to this fictitious instant of the exposure.

The plates are measured on a specially constructed comparator, and the readings are punched directly on IBM cards. About 10 stars near the moon are selected (from the Yale Catalogue) as reference stars, and their positions are measured twice. The position of the center of the moon is determined as follows:

We find an approximate center $\bar{O}$ of the moon (fig. 21). Using this point as origin, we measure the polar coordinates of a number of points on the limb. According to practice at the U.S. Naval Observatory, 30 points are taken at every 6° along the bright limb.

Using the measured polar coordinates (θ, ρ), we adjust by the method of least squares for the coordinates (x, y) of the center of the best fitted circle, as well as for its radius R.

The observation equations will be

$$\sin \theta_i x + \cos \theta_i y + R = \rho_i,$$

[13] For the I.G.Y., special telescopes were made with an aperture of 12" and a focal distance of 180".

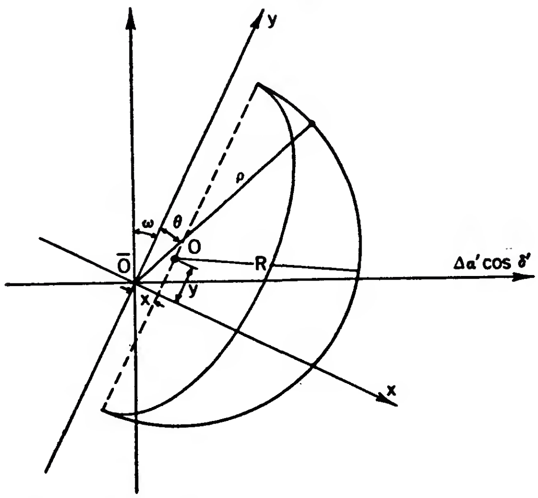

FIGURE 21.—Determination of the position of the center of the moon.

and thus the coefficient matrix for the normal equations is:

$$N=\begin{pmatrix} [\sin^2\theta] & [\sin\theta\cos\theta] & [\sin\theta] \\ [\sin\theta\cos\theta] & [\cos^2\theta] & [\cos\theta] \\ [\sin\theta] & [\cos\theta] & \nu \end{pmatrix}, \tag{159}$$

where ν is the number of measured points. For the method used by the U.S. Naval Observatory, we thus have with sufficient approximation:

$$N=\begin{pmatrix} 15 & 0 & 20 \\ 0 & 15 & 0 \\ 20 & 0 & 30 \end{pmatrix}, \tag{160}$$

and thus

$$N^{-1}=\begin{pmatrix} +0.6 & 0 & -0.4 \\ 0 & +0.0667 & 0 \\ -0.4 & 0 & +0.3 \end{pmatrix}, \tag{161}$$

if we assume that the standard error of measurement of ρ is unity.

In general,

$$N=\begin{pmatrix} \frac{\nu}{2} & 0 & \frac{2\nu}{3} \\ 0 & \frac{\nu}{2} & 0 \\ \frac{2\nu}{3} & 0 & \nu \end{pmatrix}, \tag{162}$$

and

$$N^{-1}=\begin{pmatrix} \frac{18}{\nu} & 0 & -\frac{12}{\nu} \\ 0 & \frac{2}{\nu} & 0 \\ -\frac{12}{\nu} & 0 & \frac{9}{\nu} \end{pmatrix}. \tag{163}$$

As can be seen from equations (161) or (163), the determination of the center O will not be made with the same accuracy in every direction. The error in the y direction (line of terminator) will be one-third the error in the x direction.

As a result, the determination of α' and δ' will in general not be of the same weight, and there will be an additional correlation.

If ω is the position angle of the terminator (fig. 21), we find that the variance for the determination of δ_0' and $\alpha_0'\cos\delta_0'$ of the center will be that given in equation (164), where σ_ρ is the standard error of ρ.

This variance should be combined with the one given by the errors of timing, equation (46). The motion of the moon is about 0″.5 per second of time, so even if the error of timing is 0ˢ.01, the introduced errors in the positions will be only 0″.005. This means that equation (46) can be disregarded and the variance will be given by equation (164).

To evaluate σ_ρ of equation (164), we have to consider both the errors in the measurements and the errors resulting from the topography. If the standard error of measuring the distance

$$V\left\{\begin{pmatrix} \sigma_0' \\ \alpha_0'\cos\delta_0' \end{pmatrix}\right\}=\frac{\sigma_\rho^2}{\nu}\begin{pmatrix} 18\sin^2\omega+2\cos^2\omega & -16\sin\omega\cos\omega \\ -16\sin\omega\cos\omega & 18\cos^2\omega+2\sin^2\omega \end{pmatrix}, \tag{164}$$

ρ is σ and the variance of the topography is σ^2_{top} we will have the equation,

$$\sigma^2_\rho=\sigma^2+\sigma^2_{top}. \tag{165}$$

If there is no correction for the topography of the moon, σ_{top} will be equal to the RMS of the topography, which is of the order of 0".7. This value will drop to 0".2 with the use of the existing maps of the moon and perhaps to less than 0".1 after the work of Watts is completed. The value of σ, on the other hand, is expected (Markowitz, 1959a) to be of the order of 1μ on the plate. With a focal distance of 180'', this corresponds to about 0".04. This means that the errors are due mainly to insufficient knowledge of the moon's topography.

As an example, we give the following values of $\sigma_{\delta'}$, and $\sigma_{\alpha'}$, assuming the position angle of the terminator $\omega=0$ and the declination $\delta'=0$ (in this case there is no correlation):

Without correction for the topography:

$\sigma_{\delta'}=\pm0".18$, $\sigma_{\alpha'}=\pm0".54$.

With correction for the topography from existing maps:

$\sigma_{\delta'}=\pm0".05$, $\sigma_{\alpha'}=\pm0".15$.

With correction for the topography from Watts' maps (expected):

$\sigma_{\delta'}=\pm0".03$, $\sigma_{\alpha'}=\pm0".09$.

The ratio between the weights in all cases is 9:1.

There is no doubt that if the moon could be observed around all the limb, the results would be much better. In such a case there would be no correlation between δ' and $\alpha'\cos\delta'$, and the standard error of one coordinate would be $\sigma_\rho\sqrt{\frac{2}{\nu}}$.

Determination of positions and expected accuracy.—The position of a station from which we have observed the moon with a moon camera will be obtained with the same method given for a satellite of known orbit (p. 147). Since the value of the moon's mean parallax is doubtful, it would be valuable to introduce it as an additional unknown, as described on page 143 ff. We introduce an unknown scale factor $(1+\kappa)$ and we determine the value of κ.

But in this case, since we have not measured distances but only directions, it will be impossible to solve for κ. This follows immediately from a geometric consideration. Thus, the computed positions of the stations will be given within the unknown scale factor $(1+\kappa)$. If we have two stations of known distance apart, the value of κ can be obtained by comparing the computed with the known distance.

This means that to get a complete solution we must have stations geodetically connected. The introduction of the conditions will be made as on page 132 ff.

The accuracy obtainable depends on the accuracy of our observations, on the correctness of the lunar theory, and on the net configuration. A general analysis is given in page 126 ff. We will here introduce some numerical values to get a better idea.

For simplicity, we will assume that the moon is always at the same distance r and that the observed values of α' and δ' have no correlation. But we will assume that the weight of $\alpha'\cos\delta'$ is $\frac{1}{9}$ the weight of δ'. (This is the case in which the position angle of the terminator is zero.) The normal matrix will then be that shown in equation (166), on the following page.

If we have a great number s of observations, we can replace the elements of the matrix with their mean (expected) values, considering that θ' takes the values $\lambda-90°<\theta'<\lambda+90°$, δ $-30°<\delta<+30°$, and assuming that the values are randomly distributed.

We find approximately,

$$N=\frac{s}{r^2}\begin{bmatrix}0.5 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 0.9\end{bmatrix},$$

or

$$N^{-1}=\frac{r^2}{s}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1.1\end{bmatrix}. \tag{167}$$

Therefore the variance of X^i is

$$V=\frac{\sigma^2 r^2}{s}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1.1\end{bmatrix},$$

σ being the standard error of the weight unit.

$$N=\frac{1}{r^2}\begin{Bmatrix} [\cos^2\theta'\sin^2\delta']+\frac{1}{9}\left[\frac{\sin^2\theta'}{\cos^2\delta'}\right] & [\cos^2\theta'\sin\theta'\sin^2\delta']-\frac{1}{9}\left[\frac{\sin\theta'\cos\theta'}{\cos^2\delta'}\right] & -[\cos\theta'\sin\delta'\cos\delta'] \\ [\cos\theta'\sin\theta'\sin^2\delta']-\frac{1}{9}\left[\frac{\sin\theta'\cos\theta'}{\cos^2\delta'}\right] & [\sin^2\theta'\sin^2\delta']+\frac{1}{9}\left[\frac{\cos^2\theta'}{\cos^2\delta'}\right] & -[\sin\theta'\sin\delta'\cos\delta'] \\ -[\cos\theta'\sin\delta'\cos\delta'] & -[\sin\theta'\sin\delta'\cos\delta'] & [\cos^2\delta'] \end{Bmatrix}. \qquad (156)$$

The standard errors σ^i in X^i will be

$$\sigma^1=\sigma r\sqrt{\frac{2}{s}}$$

$$\sigma^2=\sigma r\sqrt{\frac{2}{s}}$$

$$\sigma^3=\sigma r\sqrt{\frac{1.1}{s}}.$$

If we make no correction for the topography we have (see p. 154) $\sigma = 0\overset{''}{.}18$, and if we have 100 observations we will get the following errors in the coordinates:

$$\sigma^1=\pm 48 \text{ meters},$$

$$\sigma^2=\pm 48 \text{ meters},$$

$$\sigma^3=\pm 36 \text{ meters}.$$

With the additional corrections for the topography from existing maps and from Watt's maps (p. 155) the errors will be respectively:

$$\sigma^1=\pm 13 \text{ meters},$$

$$\sigma^2=\pm 13 \text{ meters},$$

$$\sigma^3=\pm 10 \text{ meters}$$

and

$$\sigma^1=\pm 8 \text{ meters},$$

$$\sigma^2=\pm 8 \text{ meters},$$

$$\sigma^3=\pm 6 \text{ meters}.$$

This estimated accuracy looks very promising. Unfortunately, this is not the true accuracy obtainable, since we have neglected the systematic errors. Many efforts have been made to eliminate those errors occurring in the camera and the comparator (Markowitz, 1959a).

Both accidental and systematic errors in the positions of the stars used as reference introduce another source of error.

If the standard error of one coordinate of a star is $\pm 0\overset{''}{.}1$ and we use 10 reference stars, the computed coordinates of the moon will have an additional error of about $\pm 0\overset{''}{.}06$. This will increase the errors σ^i by some meters.

Only after the final adjustment has been made will we be able to determine the accuracy of our result. However, we may expect that the coordinates of the stations will be computed with an accuracy of the order of 30 to 50 meters.

A final source of error that we must consider is the correctness of the ephemeris of the moon. The coordinates of the moon in the *Improved Lunar Ephemeris* are given to within $0''.01$ in δ, $0^s.001$ in α, and $0''.001$ in ϖ. From this rounding we thus have errors approximately equal to ± 5 meters in δ, ± 7 meters in α, and ± 30 meters in ϖ. With a great number of observations, however, as mentioned on page 128, the errors so introduced will be very small.

What is important is the correctness of the theory of the moon from which the ephemerides were prepared, since it will introduce systematic errors. As has already been pointed out, the mean parallax of the moon, ϖ_0, is very doubtful. For this reason it will be advisable to use the mean distance [14] of the moon as an additional unknown. We will then get much better results, considering that the mean distance may be wrong by several kilometers.

It is not easy to estimate the correctness of the theory of the moon and how it will affect our solutions. On the other hand, by analyzing the residuals we should be able to check and probably correct the orbit of the moon. The same holds also for the relative position of the center of figure with respect to the center of gravity.

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Symbols used

Following is a list of symbols used and their meanings:

[14] This is affected not only by errors of the mean parallax, but also by the earth's equatorial radius.

- a, semimajor axis
- A, azimuth
- b^u, vector or orbital elements
- e, eccentricity
- E, eccentric anomaly
- H, terrestrial height
- H_G, geodetic height
- H_A, astronomic height
- J, coefficient of second harmonic of the earth's potential
- k_e^2, geocentric gravitational constant
- l^i, direction cosines in the sidereal system
- L, geometric distance between the two stations
- m^i, direction cosines in the instantaneous terrestrial system
- M, mean anomaly
- n^i, direction cosines in the terrestrial system
- n, mean anomalistic motion
- N^j_i, matrix of normal equations
- P^j_i, weight matrix
- q^j_i, direction cosines in the geodetic system
- q, perigee distance
- r, topocentric distance
- R, geocentric distance (on pp. 118 to 122, R=correction for refraction)
- t, Greenwich sidereal time
- T, time as independent variable
- v, true anomaly
- V, variance
- W^i, coordinates in mean sidereal system
- x^i, coordinates in geodetic system
- X^i, coordinates in terrestrial system
- Y^i, coordinates in instantaneous terrestrial system
- Z^i, coordinates in apparent sidereal system
- Z, zenith distance
- α, right ascension, geocentric
- α', right ascension, topocentric
- ♈, equinox
- δ, declination, geocentric
- δ', declination, topocentric
- ζ, $H_A - H$
- η, deflection of the vertical in prime vertical
- θ, $\alpha - t$
- λ, terrestrial longitude
- λ_G, geodetic longitude
- λ_A, astronomic longitude
- ξ, deflection of the vertical in meridian
- ρ, radius vector, geocentric
- σ, standard deviation
- τ, small time interval
- ω, argument of perigee
- Ω, right ascension of the node

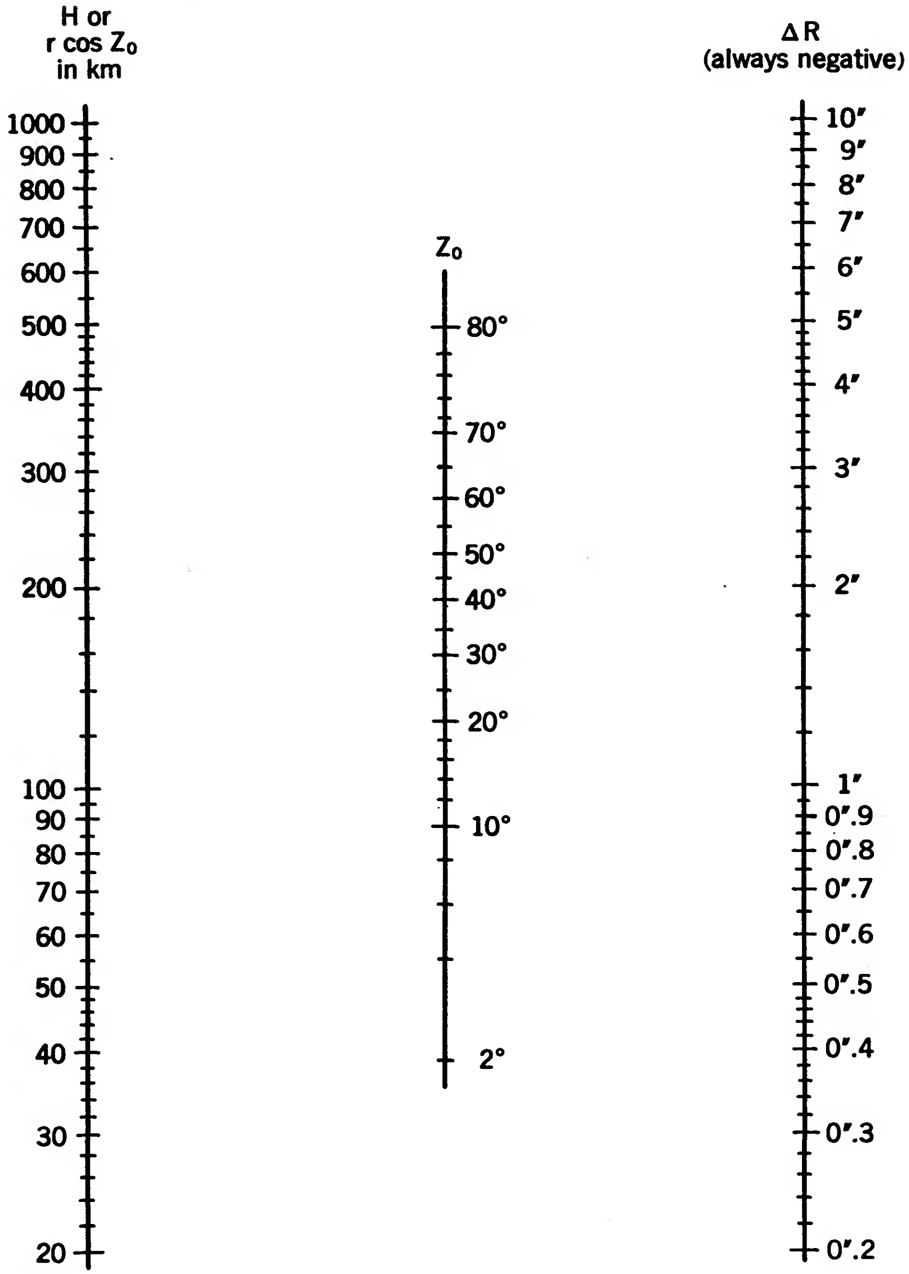

FIGURE 22.—Nomogram giving correction for refraction.

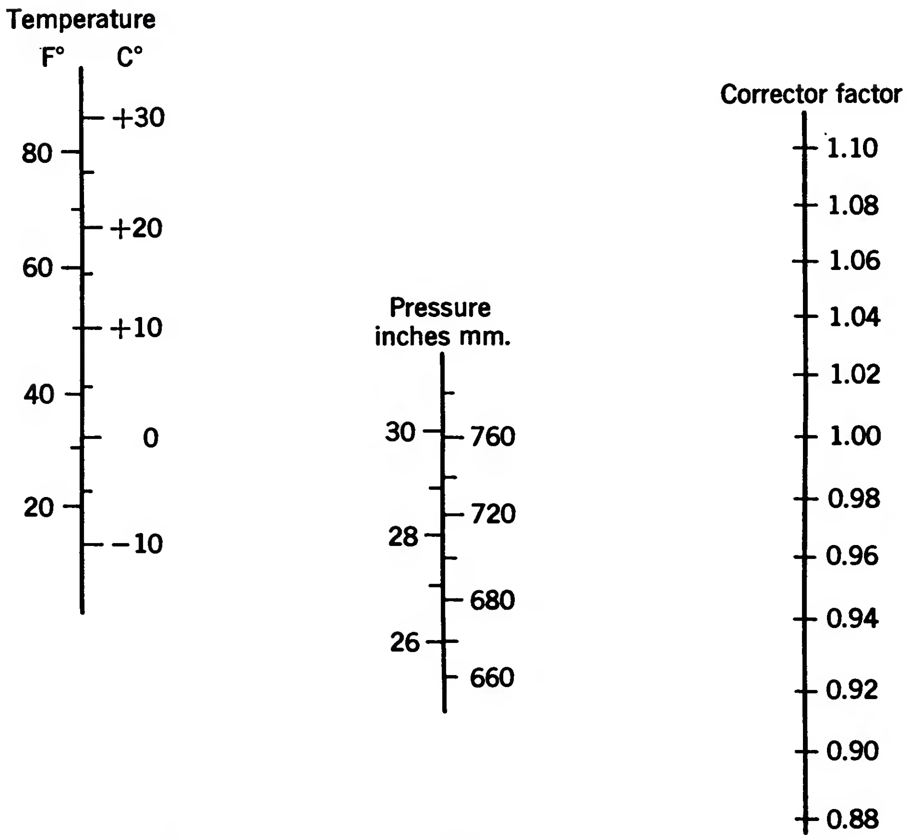

FIGURE 23.—Nomogram for correcting refraction for temperature and pressure.

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