Smithsonian

Contributions to Astrophysics

VOLUME 5, NUMBER 9

ROTATION OF AN EARTH SATELLITE IN FLIGHT ALONG ITS ORBIT

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SMITHSONIAN INSTITUTION

Washington, D.C.

1961

Publications of the Astrophysical Observatory

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Rotation of an Earth Satellite in Flight along its Orbit

By Yusuke Hagihara [1]

Introduction

The translational motion of an earth satellite as affected by radiation pressure (Musen, 1960), the electrostatic charge induced in the satellite body by the charge distribution in space, the electric current induced by the satellite's motion through the geomagnetic field (Jastrow and Pearse, 1957; Wyatt, 1960; Beard and Johnson, 1960), and the induced magnetization caused by the geomagnetic field (La Paz, 1960) have been discussed recently. The rotational motion of an earth satellite, however, has not yet been worked out in detail, except for the work of Roberson (1958) and others on the gravitational torque.

In the present paper I propose to discuss the rotational motion of an earth satellite during the period of its flight along its orbit around the earth's center, and shall base the discussion on Euler's equations of motion for the rotation of a rigid body around its center of mass. The body of the satellite is supposed to be symmetrical, not only in its external shape but also in its dynamical properties about its axis of symmetry; that is, if we denote the principal moments of inertia by A, B, C, then we have $A=B$. Although it is assumed that $C<A$ in the present discussion, which applies to a body of a long prolate spheroidal shape, the result can easily be applied to a body of a flat oblate spheroidal shape, merely by changing the sign of $A-C$, at least for the tidal torque.

The effect of tidal torque is fully discussed in the present paper for the cases of a circular and an elliptic orbit. The figure of the earth is assumed to be spherical. The instantaneous orientation of the satellite body is referred to a coordinate system that is fixed in space. It will be shown that this frame should be such that its Z-axis is directed to the north pole of the orbital plane of the satellite, and that the body makes a precessional motion of long period around this Z-axis; on this motion is superimposed the nutation of short period, of the order of the satellite's orbital period.

The hydrodynamic torque is considered on the basis of an irrotational motion for an incompressible inviscid fluid, as a rough approximation until a rigorous aerodynamic treatment is available. The solution of the equations of rotational motion, including the hydrodynamic torque, is highly complicated, because the air density varies with height above the earth's sea-level; the integration is thus carried out by expanding in powers of the orbital eccentricity.

The effect of the magnetic torque due to the induced magnetization caused by the geomagnetic field is considered. The geomagnetic field is assumed to be a dipole whose pole coincides with the earth's pole.

The radiation pressure may produce a torque on the satellite. However, if its external shape is symmetrical, the resultant of the radiation pressure acts at the center of mass of the satellite and we can expect scarcely any torque due to the radiation pressure, although the effect on the translational motion is evidently very important. The electrostatic charge induced by the charge distribution in space, and the distribution of electric current inside the satellite induced by its motion through the geomagnetic field, may also produce torques. However, if we assume a symmetrical structure of the satellite body, then these effects are considered to be insignificant. Hence we shall neglect

[1] Smithsonian Astrophysical Observatory.

here the torques due to radiation pressure, electrostatic charge, and electric current distribution inside the body.

The solution given in the present paper can be tested by photometric observations of the variation in light of an earth satellite or by ultrahigh-frequency radio echoes from an earth satellite, when we have numerical values available for the figure and structure of the satellite and its orbit, and for the variation of the orbital elements.

Equations of motion

Consider an earth satellite of long cylindrical shape whose principal moments of inertia are A, B, and C; $A=B$ and $C<A$, the C-axis being taken along the longest axis of the cylinder. The satellite is supposed to move in a field of force represented by the force function U for the torque. Denote the angular velocities around the principal axes of inertia by ω_1, ω_2, and ω_3. Then Euler's equations of motion for the rotation of the satellite around its center of mass (see, for example, Klein and Sommerfeld, 1923; Whittaker, 1927; Charbonnier, 1927; Deimel, 1950) are:

$$A\frac{d\omega_1}{dt}-(A-C)\omega_2\omega_3=-\kappa_1\omega_1+\frac{\partial U}{\partial x},\qquad A\frac{d\omega_2}{dt}+(A-C)\omega_3\omega_1=-\kappa_2\omega_2+\frac{\partial U}{\partial y},\qquad C\frac{d\omega_3}{dt}=-\kappa_3\omega_3+\frac{\partial U}{\partial z},\quad (1)$$

where $\kappa_1\omega_1$, $\kappa_2\omega_2$, $\kappa_3\omega_3$ represent the rate of retardation of the angular velocities around the three principal axes of inertia due to any cause.

Take coordinate axes of fixed direction in space with the center of mass of the satellite as the origin; for example, the equatorial axes for which the Z-axis is directed parallel to the earth's north

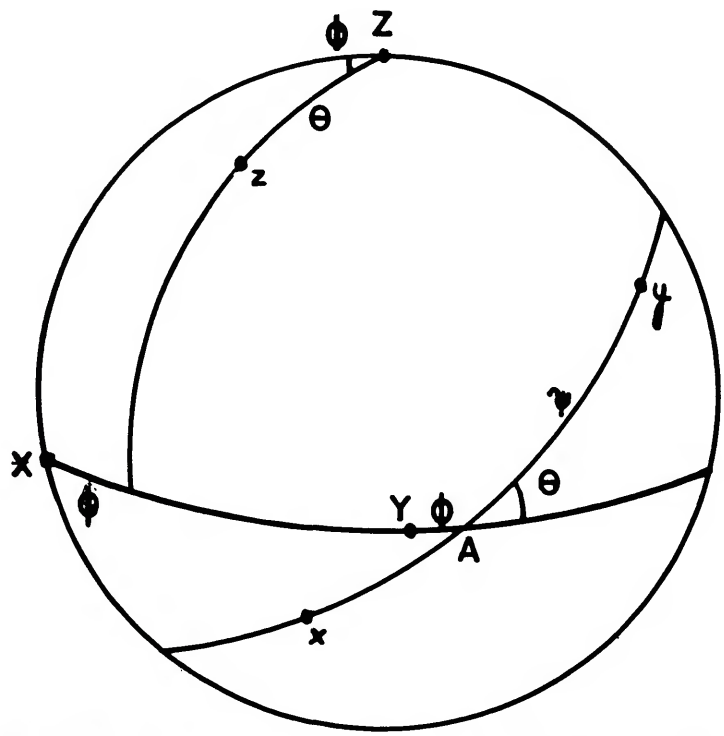

FIGURE 1.—Relation between the coordinate system (XYZ) fixed in space and the rotating coordinate system (xyz) attached to the satellite, with the Eulerian angle ($\theta\varphi\psi$).

pole, the X-axis in the plane of the earth's equator and directed to the vernal equinox, and the Y-axis in the plane of the earth's equator and perpendicular to the X-axis. Denote the Eulerian angles of the principal axes of inertia of the satellite referred to these (XYZ)-axes by θ, φ, and ψ. Then we have, as shown in table 1 (p. 142), the direction cosines of these fixed (XYZ)-axes and the rotating axes (xyz) that are attached to and carried by the body of the satellite. The x-, y-, z-axes, which are, respectively, in the direction of the A-, B-, C-axes of the principal moments of inertia of the satellite, are illustrated in figure 1.

Then we have the relations:

$$\omega_1=\frac{d\theta}{dt}\sin\psi-\frac{d\varphi}{dt}\sin\theta\cos\psi, \qquad \omega_2=\frac{d\theta}{dt}\cos\psi+\frac{d\varphi}{dt}\sin\theta\sin\psi, \qquad \omega_3=\frac{d\psi}{dt}+\frac{d\varphi}{dt}\cos\theta; \tag{2}$$

or

$$\omega_1\sin\psi+\omega_2\cos\psi=\frac{d\theta}{dt}, \qquad -\omega_1\cos\psi+\omega_2\sin\psi=\frac{d\varphi}{dt}\sin\theta. \tag{2a}$$

The force function for the tide-generating action on the satellite is expressed (Plummer, 1918) by the equation

$$U=-\frac{3}{2}\frac{m'G}{a^3}\left[(C-A)\left(\frac{z}{r}\right)^2+A\right]\left(\frac{a}{r}\right)^3, \tag{3}$$

where m' denotes the mass of the earth, G the constant of universal gravitation, a the semimajor axis of the satellite's orbit, r the radius vector of the satellite, both counted from the earth's center, and z is the rectangular coordinate in the direction of the principal axis C of the satellite. Write

$$\frac{C-A}{A}=\epsilon, \qquad \frac{3m'GA}{2a^3}=\frac{E}{\epsilon};$$

then we have,

$$U=-E\left[\left(\frac{z}{r}\right)^2+\frac{1}{\epsilon}\right]\left(\frac{a}{r}\right)^3. \tag{3a}$$

Here the (xyz)-axes are the moving coordinates attached to and carried by the satellite, and their directions coincide with its principal axes of inertia.

By substituting equation (2) in equation (1) and noting that

$$\sin\psi\frac{\partial U}{\partial x}+\cos\psi\frac{\partial U}{\partial y}=\frac{\partial U}{\partial\theta}, \qquad \cos\psi\frac{\partial U}{\partial x}-\sin\psi\frac{\partial U}{\partial y}=-\frac{\partial U}{\partial\varphi}, \qquad \frac{\partial U}{\partial z}=\frac{\partial U}{\partial\psi},$$

we get the equations of motion (1) in the form:

$$A\frac{d^2\theta}{dt^2}-A\left(\frac{d\varphi}{dt}\right)^2\sin\theta\cos\theta+C\frac{d\varphi}{dt}\sin\theta\left(\frac{d\psi}{dt}+\frac{d\varphi}{dt}\cos\theta\right)$$
$$=-\kappa_1\sin\psi\left(\frac{d\theta}{dt}\sin\psi-\frac{d\varphi}{dt}\sin\theta\cos\psi\right)-\kappa_2\cos\psi\left(\frac{d\theta}{dt}\cos\psi+\frac{d\varphi}{dt}\sin\theta\sin\psi\right)+\frac{\partial U}{\partial\theta},$$

$$A\frac{d^2\varphi}{dt^2}\sin\theta+2A\frac{d\varphi}{dt}\frac{d\theta}{dt}\cos\theta-C\frac{d\theta}{dt}\left(\frac{d\psi}{dt}+\frac{d\varphi}{dt}\cos\theta\right)=-\kappa_1\cos\psi\left(\frac{d\theta}{dt}\sin\psi-\frac{d\varphi}{dt}\sin\theta\cos\psi\right)$$
$$-\kappa_2\sin\psi\left(\frac{d\theta}{dt}\cos\psi+\frac{d\varphi}{dt}\sin\theta\sin\psi\right)+\frac{1}{\sin\theta}\frac{\partial U}{\partial\varphi},$$

$$C\left(\frac{d^2\psi}{dt^2}+\frac{d^2\varphi}{dt^2}\cos\theta-\frac{d\varphi}{dt}\frac{d\theta}{dt}\sin\theta\right)=-\kappa_3\left(\frac{d\psi}{dt}+\frac{d\varphi}{dt}\cos\theta\right)+\frac{\partial U}{\partial\psi}. \tag{4}$$

Torque due to resistance.—At first we consider the general behavior of the resisting retardation of the angular velocities due to the terms $-\kappa_1\omega_1$, $-\kappa_2\omega_2$, $-\kappa_3\omega_3$ of the equations (1). As we have supposed $C<A$, we can assume that $\kappa_1=\kappa_2>\kappa_3$. From equations (1) we have, by neglecting the force function U,

$$A\frac{d\omega_1}{dt}-(A-C)\omega_2\omega_3=-\kappa_1\omega_1, \qquad A\frac{d\omega_2}{dt}+(A-C)\omega_3\omega_1=-\kappa_2\omega_2, \qquad C\frac{d\omega_3}{dt}=-\kappa_3\omega_3. \tag{5}$$

From the third equation of (5) we get

$$\omega_3=\alpha\exp\left\{-\frac{\kappa_3}{C}t\right\}, \tag{6}$$

where α is an integration constant. Inserting this expression in (5) we have

$$A\frac{d\omega_1}{dt}-(A-C)\alpha\exp\left\{-\frac{\kappa_3}{C}t\right\}\cdot\omega_2=-\kappa_1\omega_1, \qquad A\frac{d\omega_2}{dt}+(A-C)\alpha\exp\left\{-\frac{\kappa_3}{C}t\right\}\cdot\omega_1=-\kappa_2\omega_2.$$

From these we obtain the equations:

$$\frac{A}{2}\cdot\frac{d}{dt}(\omega_1^2+\omega_2^2)=-\kappa_1(\omega_1^2+\omega_2^2), \qquad A\left(\frac{d\omega_1}{dt}\omega_2-\frac{d\omega_2}{dt}\omega_1\right)-(A-C)\alpha\exp\left\{-\frac{\kappa_3}{C}t\right\}\cdot(\omega_1^2+\omega_2^2)=0.$$

Integrating these, we have

$$(\omega_1^2+\omega_2^2)=\delta^2\exp\left\{-\frac{2\kappa_1}{A}t\right\}, \qquad \frac{\omega_1}{\omega_2}=\tan\left[\frac{\alpha C\epsilon}{\kappa_3}\exp\left\{-\frac{\kappa_3}{C}t\right\}+K\right],$$

with two integration constants δ and K. From these equations we can see that, if initially $\omega_1=\omega_2=0$, then ω_1 and ω_2 are always zero, and that ω_3 varies more rapidly than ω_1 and ω_2, as $C<A$. We obtain from these integrals the equations

$$\omega_1=\sin\left[\frac{\alpha C\epsilon}{\kappa_3}\exp\left\{-\frac{\kappa_3}{C}t\right\}+K\right]\cdot\delta\exp\left\{-\frac{\kappa_1}{A}t\right\},$$
$$\omega_2=\cos\left[\frac{\alpha C\epsilon}{\kappa_3}\exp\left\{-\frac{\kappa_3}{C}t\right\}+K\right]\cdot\delta\exp\left\{-\frac{\kappa_1}{A}t\right\}. \tag{7}$$

The position in space (θ_s,φ_s) of the axis of instantaneous rotation referred to the (XYZ)-axes is obtained from table 1 (p. 142):

$$\frac{\omega_1}{\sin\theta_s\cos\varphi_s}=\frac{\omega_2}{\sin\theta_s\sin\varphi_s}=\frac{\omega_3}{\cos\theta_s}.$$

By substituting equations (6) and (7) in these ratios we get

$$\varphi_s=\frac{\alpha C\epsilon}{\kappa_3}\exp\left\{-\frac{\kappa_3}{C}t\right\}+K, \tag{8}$$

$$\tan\theta_s=\frac{\delta}{\alpha}\exp\left\{\left(-\frac{\kappa_1}{A}+\frac{\kappa_3}{C}\right)t\right\}. \tag{9}$$

We can see from these that the polar angle θ_s of the instantaneous axis tends to $\pi/2$ for $t\to\infty$ if $(\kappa_1/A)<(\kappa_3/C)$; and tends to 0 for $t\to\infty$ if $(\kappa_1/A)>(\kappa_3/C)$. The azimuth angle φ_s of the instantaneous axis decreases constantly towards the value K, which is determined by the initial value of ω_1/ω_2.

Hydrodynamic torque.—Consider an ellipse of axes $\mathbf{c}>\mathbf{a}$ immersed in an incompressible inviscid fluid of density ρ. In a two-dimensional irrotational motion of the fluid across the ellipse with

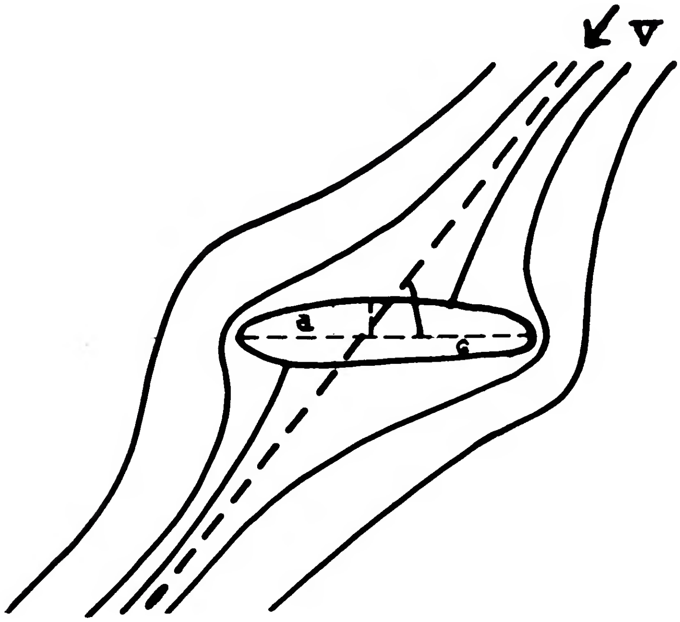

FIGURE 2.—Two-dimensional stream lines around an ellipse immersed in an incompressible fluid.

speed V at infinity, while making an angle δ with the major axis of the ellipse, the torque $\mathbf{M}$ exerted to the ellipse is, according to Cisotti (Lamb, 1932; Milne-Thomson, 1955),

$$|\mathbf{M}|=\pi\rho(\mathbf{c}^2-\mathbf{a}^2)\,V^2\sin\delta\cos\delta, \tag{10}$$

in the sense of increasing δ, or

$$|\mathbf{M}|=M\sin\delta\cos\delta, \qquad M=\pi\rho(\mathbf{c}^2-\mathbf{a}^2)\,V^2. \tag{11}$$

We extend this result to a three-dimensional motion for an ellipsoid of revolution $\mathbf{c}>\mathbf{a}=\mathbf{b}$. Then the same formula (11) can be expected to represent the torque exerted on the ellipsoid moving with speed V relative to the fluid by making an angle δ with the major axis $\mathbf{c}$ of the ellipsoid.

In the high atmosphere the air density is so low that this hydrodynamic approximation is certainly not valid. Until a rigorous aerodynamical treatment is available, based on a consideration of the detailed mechanisms involved when particles of the atmosphere impinge on the surface of the satellite, we shall tentatively assume the formula (11), with a correcting numerical factor (Rand Symposium, 1959) which I shall omit.[2]

There may be a torque due to radiation pressure. If the resultant R of the solar radiation pressure reacts at a point at a distance D from the center of mass of the satellite in the direction of its longest axis, which makes an angle σ with the direction towards the sun, then a torque of magnitude $RD\sin\sigma$ will result. However, if the structure of the satellite is symmetrical, then the resultant of the radiation pressure acts at the center of mass of the satellite and $D=0$; hence the torque vanishes, although the effect on the translational motion of the satellite is very important (Musen, 1960).

The electrostatic charge induced by the charge distribution in space also causes a drag on the motion of a satellite. The symmetrically induced charge has little effect. The induced current due to the motion across the geomagnetic field also affects the drag (Jastrow and Pearse, 1957; Wyatt, 1960; Beard and Johnson, 1960). If the satellite is symmetrically constructed, then these

[2] Personal communication from Dr. C. A. Whitney. The numerical factor is not yet available.

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effects produce an insignificant amount of torque on the satellite (Spitzer, 1960). Hence we neglect all these effects in the present paper.

Magnetic torque.—The effect of magnetic damping on the motion of a satellite has been considered by La Paz (1960) and others. I shall now consider the effect of this magnetic induction on torque.

If we denote the mean flux of the magnetic induction by **B**, the dipole moment of a ferromagnetic material placed in a magnetic field **H** is represented (Spitzer, 1960) by

$$|\mathbf{L}|=\frac{\mathbf{H}\times(\mathbf{B}-\mathbf{H})\cdot(\text{volume of the satellite})}{4\pi},$$

or

$$|\mathbf{L}|=\frac{|\mathbf{H}|\cdot|\mathbf{B}|\cdot(\text{volume of the satellite})}{4\pi}\cdot\sin\gamma,$$

where γ denotes the angle between the magnetic field **H** and the magnetic induction **B**, where $\mathbf{B}=\mathbf{H}+4\pi\mathbf{I}=\mu\mathbf{H}$ inside a ferromagnetic material (Jeans, 1940). Here **I** is the magnetization and μ the magnetic permeability. We assume that the induction is in the direction of the z-axis of the satellite, so that we have, with a constant field H_0 which will be defined later,

$$|\mathbf{B}|=\mu\left(\frac{H}{H_0}\right)\cos\gamma\cdot H_0. \tag{12}$$

With these assumptions we have

$$|\mathbf{L}|=L_0\left(\frac{H}{H_0}\right)^2\sin\gamma\cos\gamma,\qquad L_0=\frac{H_0^2\mu\cdot(\text{volume of the satellite})}{4\pi}, \tag{13}$$

the torque being in the direction of turning **B** towards **H**, that is, of decreasing γ.

The earth is supposed to be a uniformly magnetized sphere of magnetic moment M_E and magnetization I_E, where $M_E=(4\pi/3)I_E$. Then at a point with cylindrical coordinates r and θ referred to the earth's center, the radial and the tangential components of the force are poloidal and are represented by the equations

$$R=-\frac{8\pi}{3}I_E\left(\frac{r_0}{r}\right)^3\cos\theta,\qquad S=-\frac{4\pi}{3}I_E\left(\frac{r_0}{r}\right)^3\sin\theta,$$

where θ is the angular distance from the geomagnetic north pole. Here R is counted positive for the outward direction and S is positive for the direction of increasing θ, and r_0 denotes the earth's radius (Chapman and Bartels, 1951). Then we have

$$M_E=\frac{4\pi}{3}r_0^3I_E=H_0r_0^3.$$

Assume that the geomagnetic pole coincides with the earth's north pole. The components of the geomagnetic field in the directions of the (XYZ)-axes at a point of longitude λ and latitude β are

$$\frac{\mathbf{H}_X}{|\mathbf{H}|}=-\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\cos\lambda,\qquad \frac{\mathbf{H}_Y}{|\mathbf{H}|}=-\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\sin\lambda,\qquad \frac{\mathbf{H}_Z}{|\mathbf{H}|}=\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}. \tag{14}$$

In fact,

$$\mathbf{H}_Z=S\sin\theta-R\cos\theta=H_0\left(\frac{r_0}{r}\right)^3(1-3\sin^2\beta),$$

$$\mathbf{H}_R=-S\cos\theta-R\sin\theta=-H_0\left(\frac{r_0}{r}\right)^3\cdot 3\sin\beta\cos\beta,$$

$$\mathbf{H}=H_0\left(\frac{r_0}{r}\right)^3\sqrt{1+3\sin^2\beta},$$

where R is the radial direction perpendicular to the Z-axis and λ, β denote respectively the geographic longitude and latitude of the point in question at a distance r from the earth's center, under our assumption.

The first approximation.—We suppose that ω_3 is very large initially, because of the procedure in launching the satellite. We put

$$\omega_3=\frac{d\psi}{dt}+\frac{d\varphi}{dt}\cos\theta=n. \tag{15}$$

The value of n is supposed to be very large compared with ω_2 and ω_3. Thus we can treat n as a constant and set $\partial U/\partial\psi=0$, as our first approximation. We are left with two equations:

$$A\frac{d^2\theta}{dt^2}-A\left(\frac{d\varphi}{dt}\right)^2\sin\theta\cos\theta+Cn\frac{d\varphi}{dt}\sin\theta=\frac{\partial U}{\partial\theta},\qquad \frac{d}{dt}\left(A\frac{d\varphi}{dt}\sin^2\theta+Cn\cos\theta\right)=\frac{\partial U}{\partial\varphi}. \tag{16}$$

Observations of earth satellites show that n is of the order of a few seconds and that the precessional motion for $d\theta/dt$ and $d\varphi/dt$ is of the order of a few weeks (P. Zadunaisky, personal communication). Hence we assume that $d\theta/dt$ and $d\varphi/dt$ are very small compared with n; we take only those terms that contain n as a factor and neglect the remaining terms. We take thus

$$\frac{d\varphi}{dt}=\frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\theta},\qquad \frac{d\theta}{dt}=-\frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\varphi}, \tag{17}$$

where

$$U=E\left[\left(\frac{z}{r}\right)^2+\frac{1}{\epsilon}\right]\left(\frac{a}{r}\right)^3, \tag{18}$$

as the equations for determining our first approximation to the solution.

As a first approximation we solve equations (17) and then substitute these first approximate solutions into the left-hand sides of equations (16) in the terms that are not multiplied by n; we solve again for θ and φ, which appear in the terms multiplied by n, as the two new unknown functions in our second approximation. We obtain the change of n from (15) by substituting in it these second approximate values of θ and φ.

We are principally concerned with the precessional motion of the satellite, so that we eliminate short-period variations with periods of the order of the orbital period of revolution by averaging the quantities in the equations over this short period.

Equatorial circular orbits

Tidal torque.—In a circular orbit we have $r=a$ and from equation (18) we obtain

$$U=E\left(\frac{z}{a}\right)^2. \tag{19}$$

The coordinates of the earth's center relative to the center of mass of the satellite are:

$$X_E=-a\cos(v+\omega),\qquad Y_E=-a\sin(v+\omega),\qquad Z_E=0, \tag{20}$$

where v is the true anomaly and ω is the longitude of the pericenter from the node which is now taken as the X-axis. By referring to table 1 (p. 142), we have

$$z=X_E\sin\theta\cos\varphi+Y_E\sin\theta\sin\varphi+Z_E\cos\theta. \tag{21}$$

Hence:

$$\left(\frac{z}{a}\right)^2=\cos^2(v+\omega)\sin^2\theta\cos^2\varphi+\sin^2(v+\omega)\sin^2\theta\sin^2\varphi+2\sin(v+\omega)\cos(v+\omega)\sin^2\theta\sin\varphi\cos\varphi.$$

In order to obtain the precessional motion by eliminating short-period terms we take the mean value of U with respect to time t:

$$<U>=\frac{1}{2\pi}\int_0^{2\pi/\mu} U\,\mu dt=\frac{1}{2\pi}E\int_0^{2\pi/\mu}\left(\frac{z}{a}\right)^2\mu dt=\frac{E}{2}\sin^2\theta,$$

where μ denotes the mean motion in the orbital revolution. Hence the equations of motion (17) become

$$\left(\frac{d\theta}{dt}\right)_1=0,\qquad \left(\frac{d\varphi}{dt}\right)_1=E_1\cos\theta, \tag{22}$$

where $E_1=\frac{E}{Cn}$, and $E_1>0$ for $C<A$, $E_1<0$ for $C>A$.

The solution of (22) is

$$\theta=\theta_0=\text{ constant},\qquad \varphi-\varphi_0=E_1\cos\theta_0\cdot(t-t_0), \tag{23}$$

where we have taken $\varphi=\varphi_0$ at $t=t_0$. Thus the z-axis of the satellite turns around the north pole by making with it a constant angle θ_0 with the angular speed $E_1 \cos\theta_0$ in the direct sense, that is, in the sense of the orbital revolution of the satellite. This is precession.

The complete equations of motion, before we take the mean value of U, are

$$\left(\frac{d\theta}{dt}\right)_1=-E_1\sin\theta\cdot\sin 2(v+\omega-\varphi),\qquad \left(\frac{d\varphi}{dt}\right)_1=E_1\cos\theta\cdot[\cos 2(v+\omega-\varphi)+1], \tag{24}$$

where $v=\mu\,(t-t_1)$, if we denote the mean motion by μ and the epoch of the pericenter passage by t_1. The z-axis of the satellite makes a nutation with the period of the orbital revolution of the satellite superimposed on this precessional motion.

Hydrodynamic torque.—According to equation (11) the hydrodynamic torque $\mathbf{M}$ is represented by $|\mathbf{M}|=M\sin\delta\cos\delta$ in the sense of increasing δ. We first compute the angle δ. For a circular motion of the satellite we have, from equation (20),

$$\frac{dX_E}{dt}=a\mu\sin(v+\omega),\qquad \frac{dY_E}{dt}=-a\mu\cos(v+\omega),\qquad \frac{dZ_E}{dt}=0. \tag{25}$$

By referring to table 1, we find the angle δ between the z-axis of the satellite and the direction of the relative velocity $\mathbf{V}$ of the atmospheric gas and the satellite, with $V=|\mathbf{V}|$:

$$\cos\delta=\left[\frac{dX_E}{dt}\sin\theta\cos\varphi+\frac{dY_E}{dt}\sin\theta\sin\varphi\right]/V=\sin\theta\sin(v+\omega-\varphi), \tag{26}$$

as

$$V^2=\left(\frac{dX_E}{dt}\right)^2+\left(\frac{dY_E}{dt}\right)^2+\left(\frac{dZ_E}{dt}\right)^2=a^2\mu^2.$$

Next we compute the direction cosines (l, m, n) of the vector $\mathbf{M}$, which is perpendicular to both the z-axis of the satellite and the velocity vector $\mathbf{V}$. They are obtained from the relation,

$$\frac{lV}{\frac{dY_E}{dt}Z_z-\frac{dZ_E}{dt}Y_z}=\frac{mV}{\frac{dZ_E}{dt}X_z-\frac{dX_E}{dt}Z_z}=\frac{nV}{\frac{dX_E}{dt}Y_z-\frac{dY_E}{dt}X_z}, \tag{27}$$

where X_z, Y_z, Z_z denote the direction cosines of the z-axis of the satellite referred to the (XYZ)-axes, which are fixed in space. The result is

$$l=-\cos(v+\omega)\cos\theta,\qquad m=-\sin(v+\omega)\cos\theta,\qquad n=\sin(v+\omega)\sin\theta\sin\varphi+\cos(v+\omega)\sin\theta\cos\varphi. \tag{28}$$

The directions of the vector **M** referred to the (xyz)-axes attached to the body of the satellite are, from table 1 (p. 142),

$$\mathbf{M}_x/|\mathbf{M}|=l(x,X)+m(x,Y)+n(x,Z)=-\cos(v+\omega)(\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)$$
$$-\sin(v+\omega)(\sin\varphi\cos\psi+\cos\theta\cos\varphi\sin\psi),$$
$$\mathbf{M}_y/|\mathbf{M}|=l(y,X)+m(y,Y)+n(y,Z)=\cos(v+\omega)(\cos\varphi\sin\psi+\cos\theta\sin\varphi\cos\psi)$$
$$+\sin(v+\omega)(\sin\varphi\sin\psi-\cos\theta\cos\varphi\cos\psi),$$
$$\mathbf{M}_z/|\mathbf{M}|=l(z,X)+m(z,Y)+n(z,Z)=0,$$

where (A, B) denotes the direction cosine of the angle between the axis represented by A and the axis represented by B. The components of **M** referred to the axes (θ, φ, ψ) are:

$$\mathbf{M}_\theta/|\mathbf{M}|=(\mathbf{M}_x\sin\psi+\mathbf{M}_y\cos\psi)/|\mathbf{M}|=\cos(v+\omega)\cos\theta\sin\varphi-\sin(v+\omega)\cos\theta\cos\varphi,$$
$$\mathbf{M}_\varphi/|\mathbf{M}|=(-\mathbf{M}_x\cos\psi+\mathbf{M}_y\sin\psi)/(|\mathbf{M}|\sin\theta)=\cos(v+\omega)\cos\varphi+\sin(v+\omega)\sin\varphi,$$
$$\mathbf{M}_\psi/|\mathbf{M}|=0.$$

Thus we have the part due to the hydrodynamic torque in the equations of motion by substituting these expressions together with (11) and (26) in equation (17):

$$\left(\frac{d\theta}{dt}\right)_2=-M\sin\theta\cos\theta\sin^2(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)},$$
$$\left(\frac{d\varphi}{dt}\right)_2=M\sin(v+\omega-\varphi)\cos(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)}. \tag{29}$$

If we put together the part due to the tidal torque and that due to the hydrodynamic torque, we get the equations:

$$\frac{d\theta}{dt}=-E_1\sin\theta\sin 2(v+\omega-\varphi)-M\sin\theta\cos\theta\sin^2(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)},$$
$$\frac{d\varphi}{dt}=E_1\cos\theta\cdot[\cos 2(v+\omega-\varphi)+1]+M\sin(v+\omega-\varphi)\cos(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)}. \tag{30}$$

Take the mean values of the right-hand members with regard to t, that is, with regard to v from 0 to 2π, since we are dealing with the case of circular motion; then

$$\langle\sin 2(v+\omega-\varphi)\rangle=0,$$
$$\langle\cos 2(v+\omega-\varphi)\rangle=0,$$
$$\langle\sin^2(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)}\rangle=\frac{2\sin^2\theta-1}{3\sin^2\theta}\cdot\mathbf{E}(\sin\theta)+\frac{\cos^2\theta}{3\sin^2\theta}\mathbf{F}(\sin\theta),$$
$$\langle\sin(v+\omega-\varphi)\cos(v+\omega-\varphi)\sqrt{1-\sin^2\theta\sin^2(v+\omega-\varphi)}\rangle=0,$$

where **F** and **E**, respectively, denote the complete elliptic integrals of the first and the second kinds (de Haan, 1858):

$$\mathbf{F}(\sin\theta)=\int_0^{\pi/2}\sqrt{1-\sin^2\theta\sin^2 x}\cdot dx,\qquad \mathbf{E}(\sin\theta)=\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2\theta\sin^2 x}}.$$

Thus we get the equations of motion:

$$\frac{d\theta}{dt}=-M\sin\theta\cos\theta\left\{\frac{2\sin^2\theta-1}{3\sin^2\theta}\mathbf{E}(\sin\theta)+\frac{\cos^2\theta}{3\sin^2\theta}\mathbf{F}(\sin\theta)\right\},$$

$$\frac{d\varphi}{dt}=E_1\cos\theta. \tag{31}$$

In order to solve these equations we expand the elliptic integrals in powers of $\sin^2\theta$, by assuming that $\sin\theta\ll 1$:

$$\mathbf{E}(\sin\theta)=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\binom{\frac{1}{2}}{m}(\sin^2\theta)^m=\frac{\pi}{2}\cdot\left(1-\frac{1}{4}\sin^2\theta-\frac{3}{64}\sin^4\theta+\ldots\right),$$

$$\mathbf{F}(\sin\theta)=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\binom{-\frac{1}{2}}{m}(\sin^2\theta)^m=\frac{\pi}{2}\left(1+\frac{1}{4}\sin^2\theta+\frac{9}{64}\sin^4\theta+\ldots\right).$$

The first equation of (31) takes the form

$$-\frac{d\theta}{Mdt}=\sin\theta\cos\theta\cdot\left\{\frac{1}{2}-\frac{1}{4}\sin^2\theta+\ldots\right\}.$$

After integration for a small $\sin^2\theta$ we get approximately,

$$\tan\theta=\tan\theta_0\cdot\exp\left\{-\frac{M}{2}(t-t_0)\right\}, \tag{32}$$

where we suppose that $\theta=\theta_0$ for $t=t_0$. By substituting (32) in the second equation of (31) and integrating it we get, approximately,

$$\varphi-\varphi_0=E_1(t-t_0)+\frac{\tan^2\theta_0}{2M}\exp\{-M(t-t_0)\}\approx E_1\left(1-\frac{1}{2}\tan^2\theta_0+\ldots\right)(t-t_0), \tag{33}$$

where we take $\varphi=\varphi_0$ for $t=t_0$.

The precessional motion of the z-axis of the satellite due to the tidal torque alone is modified by the addition of the hydrodynamic torque in such a way that the tangent of the angle θ decreases exponentially and the speed of the rotation of the plane (Zz) changes from $E_1\cos\theta_0$ to $E_1\left(1-\frac{1}{2}\tan^2\theta_0+\ldots\right)$. Thus the satellite finally tends to direct its longest axis towards a direction perpendicular to the velocity of the satellite relative to the earth's atmosphere.

Magnetic torque.—As the orbital plane is supposed to be the equator, the direction of the geomagnetic field is always in the Z-direction and hence $\gamma=\theta$. We have

$$\mathbf{L}=-L_0\left(\frac{r_0}{a}\right)^6\sin\theta\cos\theta,\qquad \mathbf{L}_x=-|\mathbf{L}|\sin\theta\sin\psi,$$

$$\mathbf{L}_y=-|\mathbf{L}|\sin\theta\cos\psi,\qquad \mathbf{L}_z=-|\mathbf{L}|\sin\theta\cos\theta\sin(\lambda-\varphi),$$

and

$$\mathbf{L}_\theta=-|\mathbf{L}|\sin\theta,\qquad \mathbf{L}_\varphi=0,\qquad \mathbf{L}_\psi=-|\mathbf{L}|\sin\theta\sin(\lambda-\varphi).$$

Hence

$$\left(\frac{d\theta}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta,\qquad \left(\frac{d\varphi}{dt}\right)_3=0,\qquad \left(\frac{d\psi}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos^2\theta\sin(\lambda-\varphi). \tag{34}$$

The solution is:

$$\varphi=\varphi_0=\text{constant},$$

$$\frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)-\frac{1}{\sin\theta}-\frac{1}{2}\log\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)+\frac{1}{\sin\theta_0}=-L_0\left(\frac{r_0}{a}\right)^6(t-t_0). \tag{35}$$

The angle θ decreases from θ_0 for $t=t_0$, while φ is kept constant.

When we consider together the tidal, the hydrodynamical, and the magnetic torques, we obtain, by averaging over the short period,

$$\frac{d\theta}{dt}=-M\sin\theta\cos\theta\left\{\frac{2\sin^2\theta-1}{3\sin^2\theta}\,\mathrm{E}\,(\sin\theta)+\frac{\cos^2\theta}{3\sin^2\theta}\,\mathrm{F}\,(\sin\theta)\right\}-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta,$$

$$\frac{d\varphi}{dt}=E_1\cos\theta. \tag{36}$$

If we suppose that $\sin\theta\ll 1$, then equation (36) reduces to

$$\frac{d\theta}{dt}=-M\sin\theta\cos\theta\cdot\left(\frac{1}{2}-\frac{1}{4}\sin^2\theta\right)-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta,$$

$$\frac{d\varphi}{dt}=E_1\cos\theta.$$

The solution is, approximately:

$$\tan\theta=\tan\theta_0\cdot\exp\left\{-\frac{M}{2}(t-t_0)\right\},$$

$$\varphi-\varphi_0=E_1\left(1-\frac{1}{2}\tan^2\theta_0+\ldots\right)(t-t_0)+\ldots, \tag{37}$$

and the effect of the magnetic torque is small compared with that of the hydrodynamical torque, if M and $L_0(r_0/a)^6$ are of the same order of magnitude.

Choice of the reference axes.—It is questionable whether we may take our reference axes (XYZ) fixed in space in such a way that the Z-axis is directed towards the pole of the equator, because there exists no invariable plane in the rotation of a satellite that is not in a free rotation; therefore, we now examine the question by taking the Z-axis to be in the direction of the initial angular momentum of the satellite (Beletskii, 1959). Let the Z-axis be in the direction of the velocity of the satellite at the pericenter $v=0$. For convenience, we take $\omega=0$. Then

$$X_E=0,\qquad Y_E=-a\cos v,\qquad Z_E=-a\sin v,$$

and

$$z=-a\cos v\sin\theta\sin\varphi-a\sin v\cos\theta,\qquad \left\langle\left(\frac{z}{a}\right)^2\right\rangle=\frac{1}{2}\sin^2\theta\sin^2\varphi+\frac{1}{2}\cos^2\theta.$$

The equations of motion take the form

$$\frac{d\theta}{dt}=-E_1\sin\theta\sin\varphi\cos\varphi,\qquad \frac{d\varphi}{dt}=-E_1\cos\theta\cos^2\varphi.$$

In order to solve these equations we suppose that $\sin\theta\neq 0$, $\cos\varphi\neq 0$. Then we get

$$\frac{\cos\theta}{\sin\theta}\frac{d\theta}{dt}=\frac{\sin\varphi}{\cos\varphi}\frac{d\varphi}{dt},$$

or, $\sin\theta\cdot\cos\varphi=\text{constant}$.

This integral shows that the angle between the equatorial pole, which is now the X-axis, and the z-axis of the satellite is constant, in accord with the result obtained on pages 120–122; that is, the equatorial pole is the pole of the precession. Hence we should take the north pole of the equator as the fixed Z-axis in space to which the rotational motion of the satellite moving in an equatorial orbit should be referred. It will be shown in the next section that the pole of the orbital plane should be taken as the Z-axis fixed in space to which the rotational motion of the satellite is to be referred, and it is the pole of the precessional motion.

Inclined circular orbits

Tidal torque.—Let the inclination of the orbital plane of the satellite to the earth's equator be I and the longitude of the node be Ω. The coordinates (XYZ), referred to the axes parallel to the earth's north pole and its equator, of the center of the earth relative to the center of mass of the satellite are:

$$\begin{aligned} X_E &= -a\{\cos (v+\omega) \cos \Omega - \sin (v+\omega) \sin \Omega \cos I\}, \\ Y_E &= -a\{\cos (v+\omega) \sin \Omega + \sin (v+\omega) \cos \Omega \cos I\}, \\ Z_E &= -a \sin (v+\omega) \sin I. \end{aligned} \tag{38}$$

Thus we have as before,

$$z = X_E \sin \theta \cos \varphi + Y_E \sin \theta \sin \varphi + Z_E \cos \theta,$$

$$U = E_1 \left(\frac{z}{a}\right)^2.$$

The mean value of $(z/a)^2$ is obtained as follows:

$$\begin{aligned} \left\langle\left(\frac{z}{a}\right)^2\right\rangle = \frac{1}{2} (\cos^2 \Omega + \sin^2 \Omega \cos^2 I) \sin^2 \theta \cos^2 \varphi + \frac{1}{2} (\cos^2 \Omega \cos^2 I + \sin^2 \Omega) \sin^2 \theta \sin^2 \varphi \\ + \frac{1}{2} \sin^2 I \cos^2 \theta + \sin^2 I \sin \Omega \cos \Omega \sin^2 \theta \sin \varphi \cos \varphi \\ + \sin I \cos I \cos \Omega \sin \theta \cos \theta \sin \varphi - \sin I \cos I \sin \Omega \sin \theta \cos \theta \cos \varphi. \end{aligned}$$

By substituting this expression in equation (17), and after some algebraic computation, we get

$$\begin{aligned} -\frac{1}{E_1} \frac{d\theta}{dt} &= \frac{1}{2} \sin^2 I \sin (2\Omega - 2\varphi) \sin \theta + \sin I \cos I \cos (\Omega - \varphi) \cos \theta, \\ -\frac{1}{E_1} \frac{d\varphi}{dt} &= -\cos^2 I \cos \theta + \sin^2 I \sin^2 (\Omega - \varphi) \cos \theta + \sin I \cos I \sin (\Omega - \varphi) \cdot (\cos^2 \theta - \sin^2 \theta)/\sin \theta. \end{aligned} \tag{39}$$

Now we can carry out the operation,

$$[-\cos I \sin \theta + \sin I \cos \theta \sin (\Omega - \varphi)] \frac{d\theta}{dt} - \sin I \sin \theta \cos (\Omega - \varphi) \frac{d\varphi}{dt},$$

by substituting the values of $d\theta/dt$ and $d\varphi/dt$ as represented on the right-hand sides of the above differential equations; the result is zero. Hence we have an integral,

$$\cos I \cos \theta + \sin I \sin \theta \sin (\Omega - \varphi) = \text{constant}.$$

By referring to the spherical triangle ZPz shown in figure 3, in which P is the pole of the orbital plane, Z is the earth's north pole, and z is the direction of the z-axis of the satellite, we see that

$$\cos \widehat{Pz} = \cos I \cos \theta + \sin I \sin \theta \sin (\Omega - \varphi) = \text{constant}. \tag{40}$$

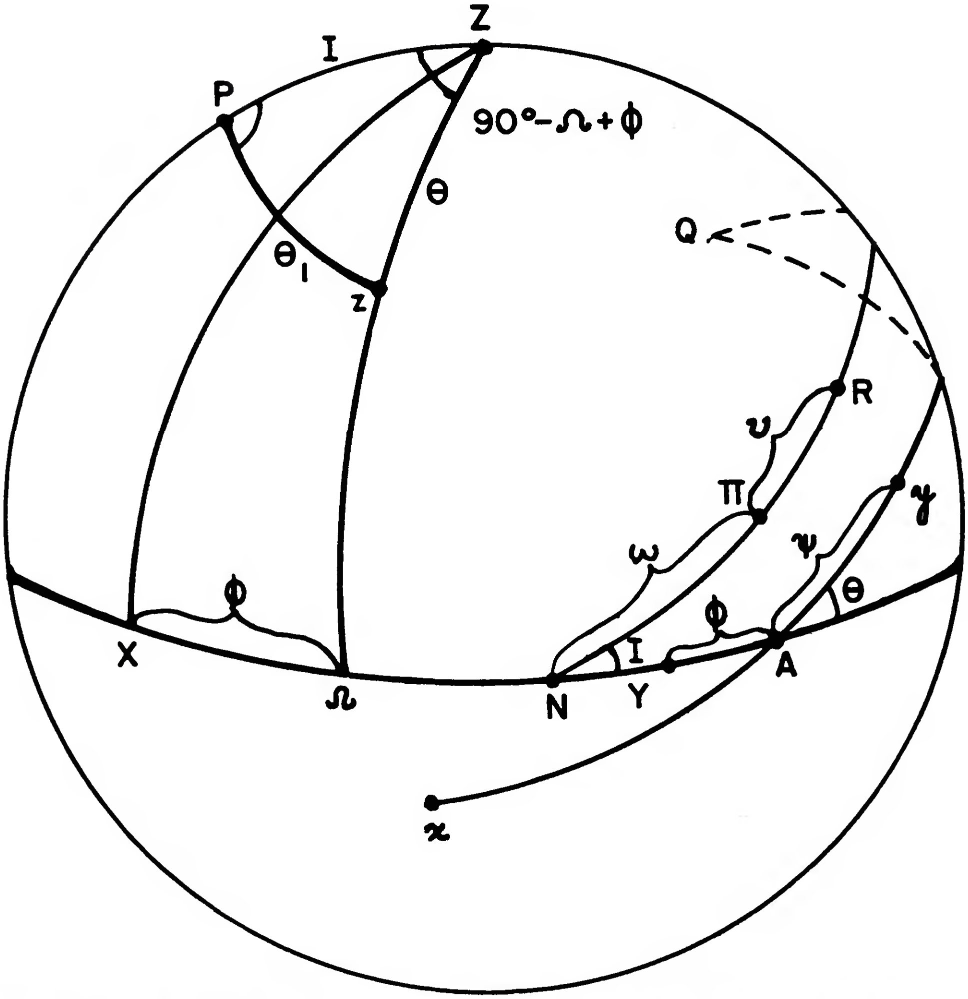

FIGURE 3.—Relation among the coordinate system (XYZ), fixed in space, the rotating coordinate system (xyz) attached to the satellite, and the orbital plane NR and its pole P.

This equation shows that *the pole of the orbital plane should be taken as the Z-axis fixed in space to which the rotational motion of the satellite is to be referred, and that it is indeed the pole of the precessional motion.*

As a special case let us consider a polar orbit; that is, an orbit with $I=90°$. Then we have

$$X_E=-a\cos(v+\omega)\cos\Omega,\qquad Y_E=-a\sin(v+\omega)\sin\Omega,\qquad Z_E=-a\sin(v+\omega),$$

$$\left\langle\left(\frac{z}{a}\right)^2\right\rangle=\frac{1}{2}\cos^2\Omega\sin^2\theta\cos^2\varphi+\frac{1}{2}\sin^2\Omega\sin^2\theta\sin^2\varphi+\frac{1}{2}\cos^2\theta+\sin\Omega\cos\Omega\sin^2\theta\sin\varphi\cos\varphi.$$

For simplicity, take $\Omega=0$; then the equations of motion are

$$\frac{1}{E_1}\frac{d\theta}{dt}=\sin\theta\sin\varphi\cos\varphi,\qquad \frac{1}{E_1}\frac{d\varphi}{dt}=-\cos\theta\sin^2\varphi.$$

We get, as before, the integral: $\sin\theta\sin\varphi=$constant. It shows that the angle between the z-axis of the satellite and the Y-axis, which is now the pole of the orbital plane, is constant. Again, we

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have seen that the pole of the orbital plane should be taken as the Z-axis fixed in space to which the rotational motion of the satellite is to be referred, and that it is indeed the pole of the precessional motion.

Hydrodynamic torque.—We take the earth's north pole as the Z-axis. We have

$$-X_E/a=\cos(v+\omega)\cos\Omega-\sin(v+\omega)\sin\Omega\cos I,$$
$$-Y_E/a=\cos(v+\omega)\sin\Omega+\sin(v+\omega)\cos\Omega\cos I,$$
$$-Z_E/a=\sin(v+\omega)\sin I, \tag{41}$$

and

$$-\frac{1}{\mu a}\frac{dX_E}{dt}=-\sin(v+\omega)\cos\Omega-\cos(v+\omega)\sin\Omega\cos I,$$
$$-\frac{1}{\mu a}\frac{dY_E}{dt}=-\sin(v+\omega)\sin\Omega+\cos(v+\omega)\cos\Omega\cos I,$$
$$-\frac{1}{\mu a}\frac{dZ_E}{dt}=\cos(v+\omega)\sin I,$$
$$V^2=\left(\frac{dX_E}{dt}\right)^2+\left(\frac{dY_E}{dt}\right)^2+\left(\frac{dZ_E}{dt}\right)^2=\mu^2a^2. \tag{42}$$

We get from equation (27), page 120, the equations:

$$l=\{\sin(v+\omega)\sin\Omega-\cos(v+\omega)\cos\Omega\cos I\}\cos\theta+\cos(v+\omega)\sin I\sin\theta\sin\varphi,$$
$$m=-\cos(v+\omega)\sin I\sin\theta\cos\varphi-\{\sin(v+\omega)\cos\Omega+\cos(v+\omega)\sin\Omega\cos I\}\cos\theta,$$
$$n=\{\sin(v+\omega)\cos\Omega+\cos(v+\omega)\sin\Omega\cos I\}\sin\theta\sin\varphi$$
$$-\{\sin(v+\omega)\sin\Omega-\cos(v+\omega)\cos\Omega\cos I\}\sin\theta\cos\varphi, \tag{43}$$

and from equation (26) we obtain

$$\cos\delta=\sin(v+\omega)\cos(\Omega-\varphi)\sin\theta+\cos(v+\omega)\{\sin(\Omega-\varphi)\sin\theta\cos I-\cos\theta\sin I\}. \tag{44}$$

We proceed in the same way as in pages 120 to 122.

$$\frac{\mathbf{M}_x}{|\mathbf{M}|}=\sin(v+\omega)\{\sin\Omega(\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta)-\cos\Omega(\sin\varphi\cos\psi+\cos\varphi\sin\psi\cos\theta)\}$$
$$+\cos(v+\omega)\{-\sin\Omega(\sin\varphi\cos\psi+\cos\varphi\sin\psi\cos\theta)+\cos\Omega(\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta)\}\cos I$$
$$-\cos(v+\omega)\sin\theta\sin\psi\sin I,$$

$$\frac{\mathbf{M}_y}{|\mathbf{M}|}=\sin(v+\omega)\{-\sin\Omega(\cos\varphi\sin\psi+\sin\varphi\cos\psi\cos\theta)+\cos\Omega(\sin\varphi\sin\psi-\cos\varphi\cos\psi\cos\theta)\}$$
$$+\cos(v+\omega)\{\sin\Omega(\sin\varphi\sin\psi-\cos\varphi\cos\psi\cos\theta)+\cos\Omega(\cos\varphi\sin\psi+\sin\varphi\cos\psi\cos\theta)\}\cos I$$
$$-\cos(v+\omega)\sin\theta\cos\psi\sin I,$$

$$\frac{\mathbf{M}_z}{|\mathbf{M}|}=\sin(v+\omega)\sin(\Omega-\varphi)\sin\theta\cos\theta,$$

and

$$\frac{\mathbf{M}_\theta}{|\mathbf{M}|} = -\sin(v+\omega)\cos(\Omega-\varphi)\cos\theta - \cos(v+\omega)\{\sin(\Omega-\varphi)\cos\theta\cos I + \sin\theta\sin I\},$$

$$\frac{\mathbf{M}_\varphi}{|\mathbf{M}|} = \{-\sin(v+\omega)\sin(\Omega-\varphi) + \cos(v+\omega)\cos(\Omega-\varphi)\cos I\}/\sin\theta,$$

$$\frac{\mathbf{M}_\psi}{|\mathbf{M}|} = \sin(v+\omega)\sin(\Omega-\varphi)\sin\theta\cos\theta.$$

Hence we get

$$\left(\frac{d\theta}{dt}\right)_2 = -M\sin\delta\cos\delta\cdot[-\sin(v+\omega)\cos(\Omega-\varphi)\cos\theta - \cos(v+\omega)\{\sin(\Omega-\varphi)\cos\theta\cos I + \sin\theta\sin I\}],$$

$$\left(\frac{d\varphi}{dt}\right)_2 = M\frac{\sin\delta\cos\delta}{\sin\theta}[-\sin(v+\omega)\sin(\Omega-\varphi) + \cos(v+\omega)\cos(\Omega-\varphi)\cos I],$$

$$\left(\frac{d\psi}{dt}\right)_2 = M\sin\delta\cos\delta\cdot\sin(v+\omega)\sin(\Omega-\varphi)\sin\theta\cos\theta. \tag{45}$$

Since

$$\cos\widehat{Pz} = \cos I\cos\theta + \sin I\sin\theta\sin(\Omega-\varphi),$$
$$\sin\widehat{Pz}\cos\angle ZPz = \sin I\cos\theta - \cos I\sin\theta\sin(\Omega-\varphi),$$
$$\sin\widehat{Pz}\sin\angle ZPz = \sin\theta\cos(\Omega-\varphi),$$

we see, by referring to figure 3, that

$$\cos\delta = -\sin\widehat{Pz}\cos(v+\omega+\angle ZPz)$$

and that

$$\frac{d\theta_1}{dt} = -E_1\sin\theta_1\cdot\sin 2(v+\omega-\varphi_1) - M\sin\theta_1\cos\theta_1\sin^2(v+\omega-\varphi_1)\sqrt{1-\sin^2\theta_1\sin^2(v+\omega-\varphi_1)},$$

$$\frac{d\varphi_1}{dt} = E_1\cos\theta_1\cdot[\cos 2(v+\omega-\varphi_1)+1] + M\sin(v+\omega-\varphi_1)\cos(v+\omega-\varphi_1)\sqrt{1-\sin^2\theta_1\sin^2(v+\omega-\varphi_1)}, \tag{46}$$

where $\theta_1 = \widehat{Pz}$ and $\varphi_1 = \angle ZPz$. Hence the equations can be treated in the same way as in (30), by taking the pole of the orbital plane as the new Z-axis fixed in space for referring the rotational motion of the satellite.

If an inclined orbit is given, we obtain the pole of the orbital plane by the equations,

$$X_P = \sin I\sin\Omega, \qquad Y_P = -\sin I\cos\Omega, \qquad Z_P = \cos I, \tag{47}$$

and we study the rotational motion of the satellite by referring to the orbital plane and its pole.

Motion of the orbital plane.—We now examine the effect of the motion of the orbital plane on the rotation of the satellite. It is known that the disturbing action causes the orbital plane of the satellite to rotate around the pole of the equator with a period of a few months (Y. Kozai, personal communication), while the orbital inclination is kept constant.

From the spherical triangle PZz shown in figure 3, where we write the angle $\angle ZPz = \varphi_1$ and the angle $\angle PZz = 90° - \Omega + \varphi$, we obtain:

$$\cos\theta = \cos I \cos\theta_1 + \sin I \sin\theta_1 \sin\varphi_1,$$

$$\sin\theta \sin(\Omega - \varphi) = -\sin\theta_1 \cos I \cos\varphi_1 + \cos\theta_1 \sin I,$$

$$\sin\theta \cos(\Omega - \varphi) = \sin\theta_1 \sin\varphi_1. \tag{48}$$

Suppose that we have studied the rotational motion of the satellite by means of the angles θ_1 and φ_1 referred to the orbital plane and its pole by the theorem given on page 125, and that we have obtained the solution in the forms of equations (32) and (33):

$$\tan\theta_1 = \tan\theta_{10} \cdot \exp\left\{-\frac{M}{2}(t - t_0)\right\}, \tag{32a}$$

$$\varphi_1 - \varphi_{10} = \overline{E} \cdot (t - t_0), \tag{33a}$$

where

$$\overline{E} = E_1\left(1 - \frac{1}{2}\tan^2\theta_{10} + \ldots\right).$$

Here we have changed the notation from θ and φ of equations (32) and (33) to θ_1 and φ_1 referred to the orbital plane and its pole. If we substitute these expressions in equations (48) and if we assume that the variation, caused by the perturbations of the translational motion of the satellite's orbital plane referred to the equator and its pole, is expressed by the relations

$$I = I_0, \qquad \Omega = \Omega_0 + \nu t, \tag{49}$$

then we obtain the equations:

$$\cos\theta = \cos I_0 \cos\theta_{10} + \sin I_0 \sin\theta_{10} \sin(\varphi_{10} + \overline{E}t) \cdot \exp\left\{-\frac{M}{2}(t - t_0)\right\},$$

$$\sin\theta \sin(\varphi - \Omega_0 - \nu t) = \sin\theta_{10} \cos I_0 \cos(\varphi_{10} + \overline{E}t) \cdot \exp\left\{-\frac{M}{2}(t - t_0)\right\} + \cos\theta_{10} \sin I_0,$$

$$\sin\theta \cos(\varphi - \Omega_0 - \nu t) = \sin\theta_{10} \sin(\varphi_{10} + \overline{E}t) \cdot \exp\left\{-\frac{M}{2}(t - t_0)\right\}.$$

By solving these equations we obtain the angles θ and φ, which show the orientation of the z-axis of the satellite referred to the equational coordinate axes (X, Y, Z), in the form:

$$\cos\theta = \cos I_0 \cos\theta_{10} + \sin I_0 \sin\theta_{10} \sin(\varphi_{10} + \overline{E}t) \cdot \exp\left\{-\frac{M}{2}(t - t_0)\right\},$$

$$\tan(\varphi - \Omega_0 - \nu t) = \left[\sin\theta_{10} \cos I_0 \cos(\varphi_{10} + \overline{E}t) + \cos\theta_{10} \sin I_0 \cdot \exp\left\{\frac{M}{2}(t - t_0)\right\}\right] / [\sin\theta_{10} \sin(\varphi_{10} + \overline{E}t)]. \tag{50}$$

The results shown in equations (32) and (33) are obtained under the assumption that $\sin\theta \ll 1$. There is no restriction on the value of I_0 in these formulas. From this result we see that the motion of the orbital plane affects only the azimuth angle φ, and not the polar angle θ, when both angles are referred to the equator and its pole.

Magnetic torque.—By combining the direction cosines of the z-axis of the satellite,

$$X_z=\sin\theta\cos\varphi,\qquad Y_z=\sin\theta\sin\varphi,\qquad Z_z=\cos\theta,$$

with the direction cosines of the magnetic field $\mathbf{H}$ of equation (14) (p. 118) we obtain:

$$\cos\gamma=-\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta\cos(\lambda-\varphi)+\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\cos\theta. \tag{51}$$

Further, we obtain:

$$-\frac{\mathbf{L}_X}{|\mathbf{L}|}=\frac{\mathbf{H}_YZ_z-\mathbf{H}_ZY_z}{|\mathbf{H}|}=\frac{-3\sin\beta\cos\beta\sin\lambda\cos\theta-(1-3\sin^2\beta)\sin\theta\sin\varphi}{\sqrt{1+3\sin^2\beta}},$$

$$-\frac{\mathbf{L}_Y}{|\mathbf{L}|}=\frac{\mathbf{H}_ZX_z-\mathbf{H}_XZ_z}{|\mathbf{H}|}=\frac{(1-3\sin^2\beta)\sin\theta\cos\varphi+3\sin\beta\cos\beta\cos\lambda\cos\theta}{\sqrt{1+3\sin^2\beta}},$$

$$-\frac{\mathbf{L}_Z}{|\mathbf{L}|}=\frac{\mathbf{H}_XY_z-\mathbf{H}_YX_z}{|\mathbf{H}|}=\frac{3\sin\beta\cos\beta\sin\theta\sin(\lambda-\varphi)}{\sqrt{1+3\sin^2\beta}};$$

$$-\frac{\mathbf{L}_x}{|\mathbf{L}|}=\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}[-\sin(\lambda-\varphi)\cos\psi+\cos\theta\cos(\lambda-\varphi)\sin\psi]+\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta\sin\psi,$$

$$-\frac{\mathbf{L}_y}{|\mathbf{L}|}=\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}[\sin(\lambda-\varphi)\sin\psi+\cos\theta\cos(\lambda-\varphi)\cos\psi]+\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta\cos\psi,$$

$$-\frac{\mathbf{L}_z}{|\mathbf{L}|}=\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta\cos\theta\sin(\lambda-\varphi);$$

$$-\frac{\mathbf{L}_\theta}{|\mathbf{L}|}=\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\cos\theta\cos(\lambda-\varphi)+\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta,$$

$$-\frac{\mathbf{L}_\varphi}{|\mathbf{L}|}=\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\frac{\sin(\lambda-\varphi)}{\sin\theta},$$

$$-\frac{\mathbf{L}_\psi}{|\mathbf{L}|}=\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}\sin\theta\cos\theta\sin(\lambda-\varphi),$$

$$|\mathbf{L}|=+L_0\left(\frac{r_0}{r}\right)^6(1+3\sin^2\beta).$$

Hence the equations of the rotational motion of the satellite are:

$$\left(\frac{d\theta}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sqrt{1+3\sin^2\beta}\cdot\sin\gamma\cos\gamma\times\{3\sin\beta\cos\beta\cos\theta\cos(\lambda-\varphi)+(1-3\sin^2\beta)\sin\theta\},$$

$$\left(\frac{d\varphi}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sqrt{1+3\sin^2\beta}\cdot\sin\gamma\cos\gamma\times3\sin\beta\cos\beta\sin(\lambda-\varphi)/\sin\theta,$$

$$\left(\frac{d\psi}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sqrt{1+3\sin^2\beta}\cdot\sin\gamma\cos\gamma\times(1-3\sin^2\beta)\sin\theta\cos\theta\sin(\lambda-\varphi). \tag{52}$$

Because of the orbital motion of the satellite the longitude λ and the latitude β, indicating the position of the satellite, vary according to the relations,

$$\sin\beta=\sin I\sin(v+\omega),\qquad \tan\lambda=\cos I\tan(v+\omega). \tag{53}$$

Put

$$\cos(\mathbf{H},\mathbf{R})=-\frac{3\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}=\sin\chi_1,\qquad \cos(\mathbf{H},\mathbf{Z})=\frac{1-3\sin^2\beta}{\sqrt{1+3\sin^2\beta}}=\cos\chi_1. \tag{54}$$

Denote the pole of the instantaneous direction of the vector **H** by H as shown in figure 4. Then $ZH=\chi_1$, $Hz=A$, $Zz=\theta$. Further denote the angles $\angle ZHz$ and $\angle HzZ$ by Θ and X, respectively, the angle $\angle HZz$ being $180°-(\lambda-\varphi)$. Then for the spherical triangle ZHz shown in figure 4 we have the following relations:

$$\cos A=\sin\chi_1\sin\theta\cos(\lambda-\varphi)+\cos\chi_1\cos\theta,$$

$$\begin{aligned}
&\sin A\cos \mathrm{X}=-\sin\chi_1\cos\theta\cos(\lambda-\varphi)+\cos\chi_1\sin\theta,\\
&\sin A\sin \mathrm{X}=\sin\chi_1\sin(\lambda-\varphi),\\
&\sin A\cos\Theta=-\cos\chi_1\sin\theta\cos(\lambda-\varphi)+\sin\chi_1\cos\theta,\\
&\sin A\sin\Theta=\sin\theta\sin(\lambda-\varphi).
\end{aligned} \tag{55}$$

By these formulas we see that $\gamma=A$ from equation (51) and that the equations (52) are transformed into

$$\begin{aligned}
&\left(\frac{d\theta}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)\sin A\cos A\cdot\sin A\cos \mathrm{X},\\
&\left(\frac{d\varphi}{dt}\right)_3=L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)\sin A\cos A\cdot\sin A\sin \mathrm{X}/\sin\theta.
\end{aligned} \tag{56}$$

From equation (55), by substituting (56) we obtain:

$$\frac{d\cdot\cos A}{dt}=-\sin A\cos\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}-\sin A\cos \mathrm{X}\cdot\frac{d\theta}{dt}-\sin\theta\sin A\sin \mathrm{X}\cdot\left(\frac{d\lambda}{dt}-\frac{d\varphi}{dt}\right),$$

$$\sin^2 A\cdot\frac{d\Theta}{dt}=\sin A\sin \mathrm{X}\cdot\frac{d\theta}{dt}+\sin\theta\sin A\cos \mathrm{X}\cdot\frac{d\varphi}{dt}+\cos A\sin\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}-\sin\theta\sin A\cos \mathrm{X}\cdot\frac{d\lambda}{dt},$$

$$\begin{aligned}
\sin^2 A\cdot\frac{d\mathrm{X}}{dt}=L_0\left(\frac{r_0}{a}\right)^6&(1+3\sin^2\beta)\sin^3 A\cos A\sin \mathrm{X}\times\left(\cos A\cos \mathrm{X}+\frac{\cos\Theta\sin\chi_1}{\sin\theta}\right)\\
&+\sin A\sin\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}-\sin\chi_1\sin A\cos\Theta\cdot\frac{d\lambda}{dt};
\end{aligned}$$

or

$$\frac{dA}{dt}=-L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)\sin^2 A\cos A+\cos\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}+\sin\theta\sin \mathrm{X}\cdot\frac{d\lambda}{dt}, \tag{57}$$

$$\sin^2 A\cdot\frac{d\Theta}{dt}=\cos A\sin\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}-\sin\theta\sin A\cos \mathrm{X}\cdot\frac{d\lambda}{dt}. \tag{58}$$

Now we can divide the satellite's motion into two parts:

I. The part of the motion around the direction H of the geomagnetic field at the position (λ,β) of the satellite, when we suppose that H and thus the satellite are instantaneously at rest; that is, when $d\lambda/dt=d\beta/dt=0$.

II. The part due to the motion of H caused by the translational motion of the satellite in longitude and latitude; that is, when $d\lambda/dt\neq 0$, $d\beta/dt\neq 0$.

Accordingly, the equations (57) and (58) split into two parts,

$$\frac{dA}{dt}=\left(\frac{dA}{dt}\right)_{\mathrm{I}}+\left(\frac{dA}{dt}\right)_{\mathrm{II}},\qquad \frac{d\Theta}{dt}=\left(\frac{d\Theta}{dt}\right)_{\mathrm{I}}+\left(\frac{d\Theta}{dt}\right)_{\mathrm{II}},$$

such that

$$\left(\frac{dA}{dt}\right)_{\mathrm{I}}=-L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)\sin^2 A\cos A,$$

$$\left(\frac{d\Theta}{dt}\right)_{\mathrm{I}}=0; \tag{59}$$

$$\left(\frac{dA}{dt}\right)_{\mathrm{II}}=\cos\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}+\sin\theta\sin\mathrm{X}\cdot\frac{d\lambda}{dt},$$

$$\left(\frac{d\Theta}{dt}\right)_{\mathrm{II}}=\cos A\sin\Theta\cdot\frac{d\chi_1}{d\beta}\frac{d\beta}{dt}-\sin\theta\sin A\cos\mathrm{X}\cdot\frac{d\lambda}{dt}. \tag{60}$$

At first we integrate the equations (59). The solution, obtained in the same way as in pages 122 to 123, is:

$$\log\left(\sqrt{\frac{1+\sin A}{1-\sin A}}\cdot\sqrt{\frac{1-\sin A_0}{1+\sin A_0}}\right)-\left(\frac{1}{\sin A}-\frac{1}{\sin A_0}\right)=-L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)(t-t_0),$$

$$\Theta=\Theta_0=\text{constant}. \tag{61}$$

Or, if $\sin A\ll 1$, the solution is, approximately,

$$\tan A=\tan A_0\cdot\exp\left\{-L_0\left(\frac{r_0}{a}\right)^6(1+3\sin^2\beta)(t-t_0)\right\},\qquad \Theta=\Theta_0. \tag{61a}$$

Hence the first part of the motion consists in an exponential shortening of the arc A, that is, the angular distance of the z-axis of the satellite from the instantaneous direction of the geomagnetic field for the point (λ,β), while the arc A is kept in the same direction fixed in space, as $\Theta=\Theta_0$ (see fig. 4). The integration of the second part of the equations is difficult and I do not go into it at present.

As the evaluation of the mean values with respect to t of the right-hand members of equation (56) is also difficult, I am now compelled to assume that $\sin\theta\ll 1$. We then have, up to the second order terms in $\sin I$, the expressions:

$$\sin\chi_1=-3\sin I\sin v,$$

$$\cos\chi_1=1-\frac{9}{2}\sin^2 I\sin^2 v,$$

$$\sin\lambda=\sin v-\frac{1}{2}\sin^2 I\sin v+\frac{1}{2}\sin^2 I\sin^3 v,$$

$$\cos\lambda=\cos v+\frac{1}{2}\sin^2 I\sin^2 v\cos v,$$

and

$$\cos A=\sin\theta\cos\varphi\sin\chi_1\cos\lambda+\sin\theta\sin\varphi\sin\chi_1\sin\lambda+\cos\theta\cos\chi_1$$

$$=\cos\theta\cdot\left[1-3\sin I\tan\theta(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)-\frac{9}{2}\sin^2 I\sin^2 v\right],$$

$$\sin A=\cos\theta\cdot\left[1+3\sin I\cot\theta\cdot(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)\right.$$
$$\left.-\sin^2 I\cdot\left(\frac{9}{2}+\frac{3}{4}\cot^2\theta\right)(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)^2+\frac{9}{2}\sin^2 I\cot^2\theta\sin^2 v\right],$$

$$\sin A\cos A=\sin\theta\cos\theta\cdot\left[1+3\sin I\cdot(\cot\theta-\tan\theta)(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)\right.$$
$$\left.-\sin^2 I\cdot\left(\frac{27}{2}+\frac{3}{4}\cot^2\theta\right)(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)^2+\frac{9}{2}\sin^2 I\cdot(\cot^2\theta-1)\sin^2 v\right],$$

$$\sin A\cos X=\sin\theta\cdot\left[1+3\sin I\cot\theta\cdot(\cos\varphi\sin v\cos v+\sin\varphi\sin^2 v)-\frac{9}{2}\sin^2 I\sin^2 v\right],$$

$$\sin A\sin X=-3\sin I\cdot[\cos\varphi\sin^2 v+\sin\varphi\sin v\cos v]+0\times\sin^2 I.$$

By substituting these on the right-hand sides of equation (56) and by taking the mean values, we get:

$$\left(\frac{d\theta}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta\cdot\left[1+\frac{3}{2}\sin I\cdot(2\cot\theta-\tan\theta)\sin\varphi\right.$$
$$\left.-\frac{9}{4}\sin^2 I\cdot(\cot^2\theta-2)-\sin^2 I\cdot\left(\frac{45}{8}-\frac{33}{16}\cot^2\theta\right)\left(1+\frac{3}{2}\sin^2\varphi\right)\right],$$

$$\left(\frac{d\varphi}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\cos\theta\cdot\left[\frac{3}{2}\sin I\cos\varphi+9\sin^2 I\cdot(\cot\theta-\tan\theta)\left(\frac{5}{8}\sin\varphi\cos\varphi\right)\right]. \tag{62}$$

At first we solve equation (62) by putting $\sin I=0$. Then we have

$$\varphi=\varphi_0=\text{constant},\qquad \frac{d\theta}{dt}=-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta;$$

or, by integrating,

$$\frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)-\frac{1}{2}\log\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)-\frac{1}{\sin\theta}+\frac{1}{\sin\theta_0}=-L_0\left(\frac{r_0}{a}\right)^6(t-t_0), \tag{63}$$

where we have taken $\theta=\theta_0$ at $t=t_0$, as before. Let us assume further that $\sin\theta\ll 1$; then we get

$$\sin\theta=\sin\theta_0-\sin^2\theta_0\cdot L_0\left(\frac{r_0}{a}\right)^6(t-t_0)+\ldots \tag{63a}$$

Now we turn to the second equation of (62) and substitute (63a) for $\sin\theta$; then we have

$$\frac{d\varphi}{dt}=-\frac{3}{2}L_0\left(\frac{r_0}{a}\right)^6\sin I\cdot\cos\theta_0\cos\varphi;$$

or, integrating from the initial value $\varphi=\varphi_0$ for $t=t_0$, we have

$$\log\left[\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right]-\log\left[\frac{1+\tan\frac{\varphi_0}{2}}{1-\tan\frac{\varphi_0}{2}}\right]=-\frac{3}{2}L_0\left(\frac{r_0}{a}\right)^6\sin I\cdot\cos\theta_0\cdot(t-t_0), \tag{64}$$

or,

$$\tan\frac{\varphi}{2}-\tan\frac{\varphi_0}{2}+\ldots=-\frac{3}{4}L_0\left(\frac{r_0}{a}\right)^6\sin I\cos\theta_0\cdot(t-t_0),$$

or,

$$\varphi=\varphi_0-\frac{3}{4}L_0\left(\frac{r_0}{a}\right)^6\sin I\cos\theta_0\cdot(t-t_0). \tag{64a}$$

We return to the first equation of (62) with these values of θ and φ given by equations (63a), (64a). Then we have the equations for the second approximation:

$$\left(\frac{d\theta}{dt}\right)_3=-L_0\left(\frac{r_0}{a}\right)^6\sin^2\theta\cos\theta\times\Big[1+3\sin I\cot\theta_0\sin\varphi_0$$
$$\times\left\{1-L_0\left(\frac{r_0}{a}\right)^6\left(\frac{3}{4}\sin I\cos\theta_0\cos\varphi_0-\sin\theta_0\right)(t-t_0)\right\}+\ldots\Big].$$

By integrating this equation we get

$$\sin\theta=\sin\theta_0-L_0\left(\frac{r_0}{a}\right)^6(\sin^2\theta_0+3\sin I\sin\theta_0\cos\theta_0\sin\varphi_0)(t-t_0)+\ldots \tag{65}$$

The formulas (64a) and (65) give the motion in θ and φ to our degree of approximation.

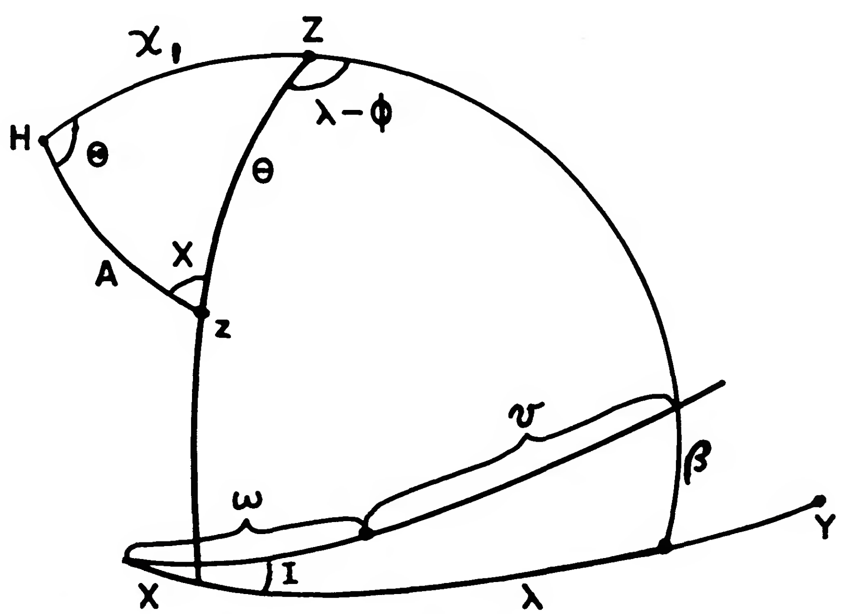

FIGURE 4.—Relation between the direction H of the geomagnetic field at a point (λ,β) on its orbit and the fixed equatorial axes (XYZ).

The formulas for θ and φ referred to the coordinate axes (XYZ) at rest in space are transformed to the coordinate axes referred to the orbital plane and its pole P by the formulas:

$$\cos\theta_1=\cos I\cos\theta+\sin I\sin\theta\sin(\Omega-\varphi),$$

$$\sin\theta_1\cos\varphi_1=-\sin\theta\cos I\sin(\Omega-\varphi)+\cos\theta\sin I,$$

$$\sin\theta_1\sin\varphi_1=\sin\theta\cos(\Omega-\varphi). \tag{66}$$

Eccentric orbits

Tidal torque.—According to the theorem which we have proved earlier (pp. 124–125), we can take the orbital plane and its pole as our reference frame (XYZ). The coordinates of the earth's center, relative to the center of mass of the satellite, are:

$$X_E=-r\cos(v+\omega),\qquad Y_E=-r\sin(v+\omega),\qquad Z_E=0. \tag{67}$$

The force function for the tidal torque is, from equation (3a), expressed in the form:

$$U=-E\left[\left(\frac{z}{r}\right)^2+\frac{1}{\epsilon}\right]\left(\frac{a}{r}\right)^3. \tag{68}$$

We have

$$\left(\frac{z}{r}\right)^2=\cos^2(v+\omega)\sin^2\theta\cos^2\varphi+\sin^2(v+\omega)\sin^2\theta\sin^2\varphi+2\sin(v+\omega)\cos(v+\omega)\sin^2\theta\sin\varphi\cos\varphi,$$

and the equations of motion are

$$\frac{d\theta}{dt}=-\frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\varphi},\qquad \frac{d\varphi}{dt}=\frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\theta}. \tag{69}$$

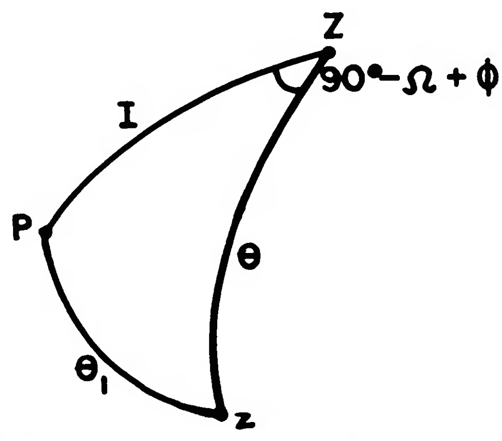

FIGURE 5.—Spherical triangle formed by the pole P of the orbital plane, the north pole Z of the equator, and the longest axis z of the satellite.

In order to eliminate the effects of short-period terms we take the mean values with respect to the mean anomaly M, by noting that

$$\frac{dv}{dM}=\sqrt{1-e^2}\left(\frac{a}{r}\right)^2, \qquad \frac{a}{r}=\frac{1}{1-e^2}(1+e\cos v). \tag{70}$$

We get

$$\left\langle\left(\frac{a}{r}\right)^3\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3\frac{dM}{dv}\,dv=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{1-e^2}}\left(\frac{a}{r}\right)dv=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+e\cos v}{(1-e^2)^{3/2}}\,dv=\frac{1}{(1-e^2)^{3/2}},$$

$$\left\langle\left(\frac{z}{r}\right)^2\left(\frac{a}{r}\right)^3\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{z}{r}\right)^2\frac{1+e\cos v}{(1-e^2)^{3/2}}\,dv=\frac{1}{(1-e^2)^{3/2}}\left(\frac{1}{2}\sin^2\theta\cos^2\varphi+\frac{1}{2}\sin^2\theta\sin^2\varphi\right)=\frac{\sin^2\theta}{2(1-e^2)^{3/2}},$$

$$\langle U\rangle=E_1\,\frac{\sin^2\theta}{2(1-e^2)^{3/2}}.$$

Hence the equations for the precessional motion are:

$$\left(\frac{d\theta}{dt}\right)_1=0, \qquad \left(\frac{d\varphi}{dt}\right)_1=\frac{E_1\cos\theta}{(1-e^2)^{3/2}}. \tag{71}$$

The solution is easily found:

$$\theta=\theta_0, \qquad \varphi-\varphi_0=\frac{E_1\cos\theta_0}{(1-e^2)^{3/2}}(t-t_0). \tag{72}$$

Hence, for an eccentric orbit, the only modification is of the factor $1/(1-e^2)^{3/2}$ in the rate of increase of φ.

Hydrodynamic torque.—The equation (11)

$$|\mathbf{M}|=M\sin\delta\cos\delta, \qquad M=\pi\rho(\mathbf{c}^2-\mathbf{a}^2)V^2,$$

is transformed to

$$|\mathbf{M}|=M_0\frac{\rho}{\rho_0}\left(\frac{V}{V_0}\right)^2\sin\delta\cos\delta, \tag{73}$$

where

$$M_0=\pi\rho_0(\mathbf{c}^2-\mathbf{a}^2)V_0^2, \qquad \rho/\rho_0=\exp\left\{-\frac{r}{r_0}\right\}, \qquad (V/V_0)^2=(a/r)^2\cdot(1-e^2\cos^2 u),$$

if r_0 denotes the radius of the earth, ρ_0 the air-density on the earth's surface, V_0 the velocity of the satellite at the pericenter, and u the eccentric anomaly in the orbital motion of the satellite. Thus we have

$$|\mathbf{M}|=M_0\cdot\exp\left\{-\frac{r}{r_0}\right\}\cdot\left(\frac{a}{r}\right)^2(1-e^2\cos^2 u)\sin\delta\cos\delta. \tag{74}$$

We propose to express the quantities in this formula by u and to take the mean values with respect to the mean anomaly M.

For simplicity we take $\omega=0$ by rotating the XY-axes by an angle ω; the X-axis is thus directed to the pericenter of the orbit. Then we have

$$X_E=-r\cos v=-a(\cos u-e), \qquad Y_E=-r\sin v=-a\sqrt{1-e^2}\cdot\sin u, \qquad Z_E=0, \tag{75}$$

and

$$\frac{dX_E}{dt}=\frac{\mu a^2}{r}\sin u, \qquad \frac{dY_E}{dt}=-\frac{\mu a^2}{r}\sqrt{1-e^2}\cdot\cos u, \qquad \frac{dZ_E}{dt}=0,$$

$$V^2=\left(\frac{dX_E}{dt}\right)^2+\left(\frac{dY_E}{dt}\right)^2+\left(\frac{dZ_E}{dt}\right)^2=\mu^2a^2\left(\frac{a}{r}\right)^2(1-e^2\cos^2 u). \tag{76}$$

The direction cosines of the angle between the velocity vector V and the (XY)-axes are

$$\cos(X,\mathbf{V})=\frac{\sin u}{\sqrt{1-e^2\cos^2 u}},\qquad \cos(Y,\mathbf{V})=-\frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}};$$

thus we get the values for the direction cosines l, m, n of equation (27):

$$l=-\frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}}\cdot\cos\theta,$$

$$m=-\frac{\sin u}{\sqrt{1-e^2\cos^2 u}}\cdot\cos\theta,$$

$$n=\frac{\sin u}{\sqrt{1-e^2\cos^2 u}}\sin\theta\sin\varphi+\frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}}\sin\theta\cos\varphi.$$

Then we have

$$\frac{\mathbf{M}_x}{|\mathbf{M}|}=-\frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}}(\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)-\frac{\sin u}{\sqrt{1-e^2\cos^2 u}}(\sin\varphi\cos\psi+\cos\theta\cos\varphi\sin\psi),$$

$$\frac{\mathbf{M}_y}{|\mathbf{M}|}=\frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}}(\cos\varphi\sin\psi+\cos\theta\sin\varphi\cos\psi)+\frac{\sin u}{\sqrt{1-e^2\cos^2 u}}(\sin\varphi\sin\psi-\cos\theta\cos\varphi\cos\psi),$$

$$\frac{\mathbf{M}_z}{|\mathbf{M}|}=0;$$

and

$$\frac{\mathbf{M}_\theta}{|\mathbf{M}|}=\frac{\cos\theta}{\sqrt{1-e^2\cos^2 u}}\cdot(\sqrt{1-e^2}\cdot\cos u\sin\varphi-\sin u\cos\varphi),$$

$$\frac{\mathbf{M}_\varphi}{|\mathbf{M}|}=\frac{1}{\sqrt{1-e^2\cos^2 u}}\cdot(\sqrt{1-e^2}\cdot\cos u\cos\varphi+\sin u\sin\varphi),$$

$$\frac{\mathbf{M}_\psi}{|\mathbf{M}|}=0,$$

$$\cos\delta=\frac{\sin\theta}{\sqrt{1-e^2\cos^2 u}}(\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi).$$

Hence our differential equations are

$$\left(\frac{d\theta}{dt}\right)_2=-M_0\exp\left\{-\frac{a}{r_0}(1-e\cos u)\right\}\cdot(1-e^2\cos^2 u)\cdot(1-e\cos u)^{-2}$$
$$\times\sin\theta\cos\theta\left(\frac{\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi}{\sqrt{1-e^2\cos^2 u}}\right)^2\cdot\left\{1-\left(\frac{\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi}{\sqrt{1-e^2\cos^2 u}}\right)^2\right\}^{1/2},$$

$$\left(\frac{d\varphi}{dt}\right)_2=M_0\exp\left\{-\frac{a}{r_0}(1-e\cos u)\right\}\cdot(1-e^2\cos^2 u)\cdot(1-e\cos u)^{-2}$$
$$\times\left(\frac{\sin u\sin\varphi+\sqrt{1-e^2}\cdot\cos u\cos\varphi}{\sqrt{1-e^2\cos^2 u}}\right)\cdot\left(\frac{\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi}{\sqrt{1-e^2\cos^2 u}}\right)$$
$$\times\left\{1-\left(\frac{\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi}{\sqrt{1-e^2\cos^2 u}}\right)^2\right\}^{1/2}. \tag{77}$$

In order to eliminate the effect of short-period terms we take the mean values of the right-hand members with respect to M by noting that

$$dM=\frac{dM}{du}\,du=\frac{r}{a}\,du=(1-e\cos u)du.$$

We make the transformation,

$$\frac{\sin u}{\sqrt{1-e^2\cos^2 u}}=\sin\chi,\qquad \frac{\sqrt{1-e^2}\cdot\cos u}{\sqrt{1-e^2\cos^2 u}}=\cos\chi, \tag{78}$$

so that

$$\frac{\sqrt{1-e^2}}{1-e^2\cos^2 u}\,du=d\chi,\qquad \frac{1}{1-e^2\cos^2 u}=\frac{\cos^2\chi}{1-e^2}+\sin^2\chi.$$

By integrating this relation between du and $d\chi$ we obtain the equation,

$$\frac{1-e}{1+e}\tan^{-1}\left(\frac{\tan\frac{u}{2}}{\sqrt{\frac{1-e}{1+e}}}\right)+\frac{1+e}{1-e}\tan^{-1}\left(\frac{\tan\frac{u}{2}}{\sqrt{\frac{1+e}{1-e}}}\right)=\chi+\text{constant}, \tag{79}$$

Or, by taking the constant equal to 0, we have

$$\chi=u\left(1+\frac{e^2}{4}+\ .\ .\ .\right),\qquad \cos u=\cos\chi+\frac{e^2}{4}\sin^2\chi+\ .\ .\ .\ . \tag{79a}$$

Further, we have

$$\frac{\sin u\cos\varphi-\sqrt{1-e^2}\cdot\cos u\sin\varphi}{\sqrt{1-e^2\cos^2 u}}=\sin(\chi-\varphi),$$

$$\frac{\sin u\sin\varphi+\sqrt{1-e^2}\cdot\cos u\cos\varphi}{\sqrt{1-e^2\cos^2 u}}=\cos(\chi-\varphi).$$

Hence we get

$$\left(\frac{d\theta}{dt}\right)_2=-\frac{M_0}{2\pi\sqrt{1-e^2}}\sin\theta\cos\theta\cdot\exp\left\{-\frac{a}{r_0}\right\}\cdot\int_0^{2\pi}\exp\left\{\frac{a}{r_0}\left(e\cos\chi+\frac{e^3}{2}\sin^2\chi+\ .\ .\ .\right)\right\}$$
$$\times\left(\frac{\cos^2\chi}{1-e^2}+\sin^2\chi\right)^{-2}\left(1-e\cos\chi-\frac{e^3}{2}\sin^2\chi+\ .\ .\ .\right)^{-1}\cdot\sin^2(\chi-\varphi)\sqrt{1-\sin^2\theta\sin^2(\chi-\varphi)}\cdot d\chi,$$

$$\left(\frac{d\varphi}{dt}\right)_2=\frac{M_0}{2\pi\sqrt{1-e^2}}\exp\left\{-\frac{a}{r_0}\right\}\cdot\int_0^{2\pi}\exp\left\{\frac{a}{r_0}\left(e\cos\chi+\frac{e^3}{2}\sin^2\chi+\ .\ .\ .\right)\right\}\times\left(\frac{\cos^2\chi}{1-e^2}+\sin^2\chi\right)^{-2}$$
$$\times\left(1-e\cos\chi-\frac{e^3}{2}\sin^2\chi+\ .\ .\ .\right)^{-1}\cdot\sin(\chi-\varphi)\cos(\chi-\varphi)\sqrt{1-\sin^2\theta\sin^2(\chi-\varphi)}\cdot d\chi. \tag{80}$$

Now we have (see p. 122)

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2(\chi-\varphi)\sqrt{1-\sin^2\theta\sin^2(\chi-\varphi)}\cdot d\chi=\frac{2\sin^2\theta-1}{3\sin^2\theta}\,\mathbf{E}(\sin\theta)+\frac{\cos^2\theta}{3\sin^2\theta}\,\mathbf{F}(\sin\theta),$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(\chi-\varphi)\sin^2(\chi-\varphi)\sqrt{1-\sin^2\theta\sin^2(\chi-\varphi)}\cdot d\chi=0,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^3(\chi-\varphi)\sqrt{1-\sin^2\theta\sin^2(\chi-\varphi)}\cdot d\chi=0.$$

But we write

$$I(s^2c^2)=\int_0^{\frac{\pi}{2}} \cos^2 x \sin^2 x\sqrt{1-\sin^2\theta\sin^2 x}\cdot dx=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\cdot\frac{1}{2}\cdot\frac{2m+1}{2m+2}\cdot\frac{2m-1}{2m}\cdot\cdots\cdot\frac{1}{2}\binom{\frac{1}{2}}{m}(\sin^2\theta)^m$$

$$=\frac{\pi}{4}\left(\frac{1}{2}-\frac{3}{16}\sin^2\theta-\frac{15}{384}\sin^4\theta\ldots\right),$$

$$I(s^4)=\int_0^{\frac{\pi}{2}} \sin^4 x\cdot\sqrt{1-\sin^2\theta\sin^2 x}\cdot dx=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\cdot\frac{1}{2}\cdot\frac{2m+3}{2m+4}\cdot\frac{2m+1}{2m+2}\cdot\cdots\cdot\frac{1}{2}\binom{\frac{1}{2}}{m}(\sin^2\theta)^m$$

$$=\frac{\pi}{4}\left(1-\frac{15}{48}\sin^2\theta-\frac{35}{512}\sin^4\theta+\ldots\right), \tag{81}$$

$$\mathbf{E}(\sin\theta)=\int_0^{\frac{\pi}{2}}(1-\sin^2\theta\sin^2 x)^{1/2}\,dx=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\,\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\cdots\cdot\frac{1}{2}\cdot\binom{\frac{1}{2}}{m}(\sin^2\theta)^m$$

$$=\frac{\pi}{2}\left(1-\frac{1}{4}\sin^2\theta-\frac{3}{64}\sin^4\theta+\ldots\right),$$

$$\mathbf{F}(\sin\theta)=\int_0^{\frac{\pi}{2}}(1-\sin^2\theta\sin^2 x)^{-1/2}\,dx=\frac{\pi}{2}\cdot\sum_{m=0}^{\infty}(-1)^m\,\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\cdots\cdot\frac{1}{2}\cdot\binom{-\frac{1}{2}}{m}(\sin^2\theta)^m$$

$$=\frac{\pi}{2}\left(1+\frac{1}{4}\sin^2\theta+\frac{9}{64}\sin^4\theta+\ldots\right). \tag{82}$$

Then we obtain the equations of the precessional motion of the satellite up to the second order of the orbital eccentricity e in the form:

$$\left(\frac{d\theta}{dt}\right)_2=-\frac{2M_0(1-e^2)^{3/2}}{\pi}\exp\left\{-\frac{a}{r_0}\right\}\cdot\sin\theta\cos\theta\times\left[\left\{\frac{2\sin^2\theta-1}{3\sin^2\theta}\mathbf{E}(\sin\theta)+\frac{1-\sin^2\theta}{3\sin^2\theta}\mathbf{F}(\sin\theta)\right\}\right.$$
$$\left.-e^2\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\left\{\cos^2\varphi\cdot I(s^2c^2)+\sin^2\varphi\cdot I(s^4)\right\}\right],$$

$$\left(\frac{d\varphi}{dt}\right)_2=E_1\cos\theta+\frac{2M_0(1-e^2)^{3/2}}{\pi}\exp\left\{-\frac{a}{r_0}\right\}\cdot\left[-e^2\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\sin\varphi\cos\varphi\cdot I(s^2c^2)\right], \tag{83}$$

where the right-hand members are also expanded in powers of $\bar{a}=a-r_0$, the height above the sea-level of the earth, up to the second order.

If we again suppose that $\sin\theta\ll 1$, then, by substituting the values in the first approximation for θ and φ from equations (32) and (33) in the terms multiplied by e^2, we have

$$\frac{2d\theta}{\sin\theta\cos\theta\cdot dt}=-M_0(1-e^2)^{3/2}\exp\left\{-\frac{a}{r_0}\right\}\cdot\left[1+\frac{1}{2}\sin^2\theta_0\cdot\exp\{-M(t-t_0)\}\right.$$
$$\left.-e^2\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\sin^2\{\varphi_0+E_1(t-t_0)\}\right],$$

$$\frac{d\varphi}{dt}=E_1\cdot\left[1-\frac{1}{2}\tan^2\theta_0\cdot\exp\{-M(t-t_0)\}\right]$$
$$+M_0(1-e^2)^{3/2}\exp\left\{-\frac{a}{r_0}\right\}\cdot\left[-\frac{e^2}{4}\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\right]. \tag{83a}$$

By integrating these equations we obtain an approximate solution:

$$\begin{aligned}\frac{\tan\theta}{\tan\theta_0}&=\exp\left[-\frac{1}{2}M_0(1-e^2)^{3/2}\exp\left\{-\frac{a}{r_0}\right\}\cdot\left\{\left(1-\frac{\sin^2\theta_0}{2}\right)-e^2\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\left(\frac{5}{8}-\frac{1}{8}\cos 2\varphi_0\right)\right.\right.\\&\qquad\left.\left.+\left(\frac{M}{2}\sin^2\theta_0-\frac{1}{4}e^2\sin 2\varphi_0\cdot\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)\right)\cdot\frac{t-t_0}{2}\right\}(t-t_0)\right],\\\frac{\tan\varphi}{\tan\varphi_0}&=\exp\left[E_1\left(1-\frac{1}{2}\tan^2\theta_0\right)\cdot(t-t_0)-\frac{1}{4}e^2M_0(1-e^2)^{3/2}\exp\left\{-\frac{a}{r_0}\right\}\cdot\left(\frac{3}{2}-\frac{\bar{a}^2}{2r_0^2}\right)(t-t_0)\right].\end{aligned}\tag{84}$$

Variations of the orbital form.—The values of the orbital semimajor axis a and the orbital eccentricity e vary slightly because of perturbations due to the figure of the earth, lunar and solar gravitational action, atmospheric drag, radiation pressure, and so forth. Suppose that the variations are expressed by the equations,

$$\begin{aligned}e&=e_0+e_1(t-t_0)+e_2(t-t_0)^2+\ldots,\\a&=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots.\end{aligned}\tag{85}$$

If we consider the effect on the tidal torque only, the effect is expressed, by use of equation (72), by the relation,

$$\begin{aligned}\theta&=\theta_0,\\\varphi-\varphi_0&=\frac{E_1\cos\theta_0}{(1-e_0^2)^{3/2}}(t-t_0)+\frac{E_1\cos\theta_0}{(1-e_0^2)^{5/2}}\cdot\left(e_0e_1-\frac{3a_1}{a_0}\right)(t-t_0)^2+\ldots.\end{aligned}\tag{86}$$

If we consider the effect on the hydrodynamic torque, the effect will be obtained by computing by use of equation (84) the increments of θ and φ due to the increments of a and e shown by equation (85).

Summary

The rotational motion of an earth satellite during its orbital flight around the earth's center is discussed on the basis of Euler's equation of motion (1) for the rotation of a rigid body around its center of mass. The body of the satellite is supposed to be symmetrical, both geometrically and dynamically, that is, with regard to its principal moments of inertia, so that $A=B$ and $C\neq A$. Especially the satellite body is supposed to have a long prolate spherical shape.

The effects are discussed of the tidal or the gravitational torque due to the action of the earth, which is supposed to be spherical; of the hydrodynamical torque caused by the satellite's motion through the earth's atmosphere; and of the magnetic torque due to the magnetic induction of the satellite material by the geomagnetic field. The radiation pressure may exert a torque, but its effect is neglected because of the symmetrical shape of the satellite body. The electrostatic charge, induced by the charge distribution in space through which the satellite moves, and the electric current distribution inside the satellite body, induced by its motion through the geomagnetic field, produce effects that are neglected for the same reason.

To study the precessional motion of the satellite, we eliminate the short-period motions, those having the period of the orbital revolution of the satellite. We summarize the results separately for the tidal torque, the hydrodynamic torque, and the magnetic torque.

1. The effect of the tidal torque is discussed on the basis of the tide-generating force exerted by the spherical earth on the satellite, in the form shown in equation (3):

$$U=-\frac{3}{2}\frac{m'G}{a^3}\left[(A-C)\left(\frac{z}{r}\right)^2+A\right]\left(\frac{a}{r}\right)^3,$$

where m' denotes the earth's mass, G the constant of gravitation, a the semimajor axis of the satellite orbit and r the radius vector of the satellite, both measured from the earth's center, and z the rectangular coordinate in the direction of the principal axis having the moment of inertia C of the satellite.

Let the instantaneous angular velocities of the satellite around its principal axes of inertia be ω_1, ω_2 and ω_3 respectively, ω_3 being around the z-axis having the moment of inertia C. Let the resistance to the rotation be supposed to be $\kappa_1\omega_1$, $\kappa_2\omega_2$, $\kappa_3\omega_3$ around each of the principal axes of inertia. We take reference axes (XYZ) fixed in space and refer the position of the principal axes of inertia of the satellite by use of the Eulerian angles θ, φ, and ψ. The angle θ, the angle between the z-axis that is attached to and rotating with the satellite and the Z-axis that is fixed in space, decreases exponentially at the rate $-\kappa_3/C$, and ω_1, ω_2 decrease exponentially at the rate $-\kappa_1/A$ as shown in equations (6) and (7). The position θ_s of the instantaneous axis of rotation for $t\rightarrow\infty$ tends to $\pi/2$ if $\kappa_1/A<\kappa_3/C$, and tends to 0 if $\kappa_1/A>\kappa_3/C$. The value of φ_s, the azimuth angle between the plane (z_s,Z) and the fixed reference plane (Z,Y), of the instantaneous axis of rotation, decreases constantly towards a constant value K determined by the initial value of ω_1/ω_2, as shown in equations (8) and (9) (p. 116).

We suppose that ω_3 is very large initially and suppose, for the first approximation, that $\omega_3=n=$ constant, just as in the case of the precession of the earth's rotation axis. As the (XYZ)-axes fixed in space we take the axes drawn from the satellite's center of mass parallel to the direction of the vernal equinox, to that of its perpendicular on the earth's equator, and to that of the north pole of the earth's equator. For an earth satellite with a circular equatorial orbit (see p. 120) we have:

$$\theta=\theta_0=\text{constant},$$

$$\varphi-\varphi_0=E_1\cdot\cos\theta_0\cdot(t-t_0),$$

where $\theta=\theta_0$, $\varphi=\varphi_0$ for $t=t_0$ and $E_1=\dfrac{3m'G(A-C)}{2a^3Cn}$.

Thus the z-axis of the satellite rotates in the sense of orbital revolution of the satellite around the pole of its orbital plane by describing a small circle of angular radius θ with the pole. This is precession. Short-period variations of the period of orbital revolution are superposed on this precession. This is nutation.

If we take our reference axes (XYZ) to be such that the Z-axis is directed towards the direction of the initial angular momentum of the satellite, then we get an integral showing that the angle between the z-axis and the earth's north pole is constant (p. 124). Hence the earth's pole is seen to be the pole of the precessional motion, and the earth's north pole should be taken as our fixed Z-axis to which we refer the rotation of the satellite moving in an equatorial orbit. The same theorem is proved for an inclined orbit; that is, the angle is constant between the z-axis attached to and rotating with the satellite and the pole of the orbital plane, and the pole of the orbital plane is the center of the precession. The theorem is proved also for a polar orbit (p. 125). Hence we see that the pole of the orbital plane should be taken as the Z-axis fixed in space to which the rotational motion of the satellite is to be referred, and it is indeed the center of the precessional motion of the satellite. This fact is proved to be true, by a suitable transformation of the coordinate axes, even if we consider the hydrodynamic torque (p. 127). Hence, if an inclined orbit is given, we compute the pole of the orbital plane by equation (47) and study the rotational motion by referring to the orbital plane and its pole.

The orbital plane of the satellite shifts because of the perturbations. If the motion of the orbital plane is given by equation (49), then the effects are shown to be those expressed by equation (50) (p. 128).

For an eccentric orbit the tidal torque causes the precession to have a minor change:

$$\theta=\theta_0,\ \varphi-\varphi_0=\frac{E_1 \cos \theta_0}{(1-e^2)^{3/2}}(t-t_0). \tag{72}$$

The semimajor axis a and the eccentricity e of the satellite orbit vary, due to perturbations; this variation is taken into account by supposing that the variation is that given by equation (85). The result is shown in equation (86) (p. 139).

2. Suppose a prolate spheroid with axes $\mathbf{c}>\mathbf{a}=\mathbf{b}$. An irrotational motion of an incompressible inviscid fluid of density ρ, moving across the spheroid with speed V at infinity, exerts a hydrodynamic torque $\mathbf{M}$ making an angle δ with the major axis $\mathbf{c}$ of the spheroid. This torque, according to Cizotti (see Lamb, 1932; Milne-Thomson, 1955), is taken to be:

$$|\mathbf{M}|=M \sin \delta \cos \delta, \qquad M=\pi\rho(\mathbf{c}^2-\mathbf{a}^2)V^2, \tag{11}$$

in the sense of increasing δ (see fig. 2, p. 117). I assume this formula for the moment, as we have no rigorous formula available for our actual case.

For an equatorial circular orbit the equations for the precessional motion of the satellite are obtained in equation (31). The hydrodynamical torque affects the precession only in θ, but not in φ. The equations are solved for $\sin \theta \ll 1$ in equations (32) and (33). The angle θ decreases exponentially and the motion in the angle φ is modified by only a small amount (p. 122). Finally, the z-axis of the satellite tends to be directed parallel to the Z-axis, that is, perpendicular to the direction of the satellite's velocity relative to the earth's atmosphere.

It is shown that the effect of the hydrodynamical torque can be properly treated by referring to the orbital plane and its pole as the fixed axes in space, and this pole is the center of the precession.

The effect on the satellite moving in an eccentric orbit is discussed (p. 135). The torque is taken to be

$$|\mathbf{M}|=M_0\left(\frac{\rho}{\rho_0}\right)\left(\frac{V}{V_0}\right)^2 \sin \delta \cos \delta, \tag{73}$$

where

$$M_0=\pi\rho_0(\mathbf{c}^2-\mathbf{a}^2)V_0^2, \qquad \frac{\rho}{\rho_0}=\exp\left\{-\frac{r}{r_0}\right\}, \qquad \left(\frac{V}{V_0}\right)^2=\left(\frac{a}{r}\right)^2(1-e^2\cos^2 u),$$

if r_0 denotes the radius of the earth, ρ_0 the air-density on the earth's surface, V_0 the velocity of the satellite at its pericenter, and u the eccentric anomaly in the orbital motion of the satellite. The equations of motion (77) are transformed by (78) to (83) by the introduction of the complete elliptic integrals (82) and the definite integrals (81). The equations are solved for the case $\sin \theta \ll 1$ by expanding in power series of e up to the square in the form (84).

3. The magnetic torque due to the induced magnetization of the satellite is taken to be

$$|\mathbf{L}|=L_0\left(\frac{H}{H_0}\right)^2 \sin \gamma \cos \gamma, \qquad L_0=\frac{H_0^2\,(\text{permeability})\times(\text{volume of the satellite})}{4\pi}, \tag{13}$$

in the sense of decreasing γ, where $H=|\mathbf{H}|$ denotes the magnetic field at the satellite position and γ the angle between the geomagnetic field $|\mathbf{H}|$ and the z-axis attached to the satellite, of which the latter is supposed to be the direction of the magnetic induction. The earth is assumed to be a uniformly magnetized sphere and the geomagnetic pole is assumed to coincide with the earth's pole. The geomagnetic field is then that of a dipole and is expressed by equation (14) (p. 119).

For an equatorial circular orbit the magnetic torque affects only the rotational motion in θ and ψ, but not in φ, as shown in equation (34). Hence the azimuth angle φ is kept constant, while θ decreases nearly linearly with t (p. 123). The effects of all three kinds of torques, the tidal, the hydrodynamical and the magnetic, are described by equation (36). The solution (37) is carried out for $\sin\theta \ll 1$. The effect of the magnetic torque is small, compared with that of the hydrodynamic torque, if M and $L_0(r_0/a)^6$ are of the same order of magnitude.

The computation of the effect of the magnetic torque on a satellite with its orbital plane inclined to the equator is very complicated. The equations of motion are deduced in the form of equation (52). It is seen that the angle $\angle ZHz = \Theta$ is constant in the spherical triangle formed by the equator's pole Z, the direction of the geomagnetic field $\mathbf{H}$ at the satellite, and the z-axis attached to the satellite (see fig. 4); and that the angle between $\mathbf{H}$ and z decreases exponentially, if we disregard the orbital motion of the satellite. This circumstance, however, does not help us in integrating our equations of motion.

We are thus compelled to assume that the inclination I is small. The equations for the precession are now in the form (62) and are solved by successive approximations (64) or (64a) and (65). To include the magnetic torque, we use the following procedure. First, we study the precessional motion caused by the tidal and hydrodynamical torques, with reference to the orbital plane and its pole, according to the theorem stated on page 125. Then we transform back to the equatorial coordinates by using equation (66). To this motion we add the precession due to the magnetic torque, supposing that the effect of the magnetic torque is sufficiently small.

TABLE 1.—*Direction cosines of fixed and rotating axes*

Rotating axes	Fixed axes		
	X	Y	Z
x	$\cos\theta\cos\varphi\cos\psi-\sin\varphi\sin\psi$	$\cos\theta\sin\varphi\cos\psi+\cos\varphi\sin\psi$	$-\sin\theta\cos\psi$
y	$-\cos\theta\cos\varphi\sin\psi-\sin\varphi\cos\psi$	$-\cos\theta\sin\varphi\sin\psi+\cos\varphi\cos\psi$	$\sin\theta\sin\psi$
z	$\sin\theta\cos\varphi$	$\sin\theta\sin\varphi$	$\cos\theta$

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Abstract

The rotational motion, especially the precessional motion, of an earth satellite around its center of mass during its orbital flight around the earth's center is discussed on the basis of Euler's equations. The body of the satellite is supposed to be symmetrical, both geometrically and dynamically. The effects of tidal and hydrodynamical torques and also of magnetic torque due to the magnetic induction in the satellite by the geomagnetic field are considered. It is proved that the pole of the orbital plane should be taken as the coordinate axis fixed in space to which the rotational motion is to be referred, and that it is indeed the pole of the precessional motion of the satellite, if the tidal and hydrodynamic torques are taken into account.

The paper begins by discussing in general the effect of torque due to resistance, and describes how the angular velocity decreases and how the instantaneous axis of rotation changes with time. The initial angular velocity around the longest principal axis of inertia is supposed to be very high, and the solution is carried out under this assumption. It is shown that the hydrodynamic torque affects the precessional motion both in the Eulerian polar angular distance θ and in the azimuth angle φ of the longest principal axis of inertia, but the tidal torque affects only φ and the magnetic torque affects only θ; and that the satellite, by performing a precessional motion around the pole of the orbital plane in the sense of orbital revolution, finally tends to direct its longest principal axis of inertia to a direction perpendicular to its orbital velocity. The total effect of all the three kinds of torques is discussed for circular and eccentric orbits of the motion of the satellite around the earth's center. For the case of an eccentric orbit, the solution is expanded in a series proceeding in powers of the orbital eccentricity. For the effect of the magnetic torque the solution is carried out only for the case of small orbital inclinations. In some cases in which the integration seems difficult the solution is obtained by expanding in powers of $\sin \theta$ by the assumption that $\sin \theta \ll 1$. The effect of the motion of the satellite's orbital plane due to perturbations is also considered.

It is hoped that the results offered in the present paper will be tested by photometric observations on the variation in brightness of satellites and by observations of ultrahigh-frequency radio echoes from satellites.

U.S. GOVERNMENT PRINTING OFFICE: 1961