Smithsonian

Contributions to Astrophysics

Volume 5, Number 6

GAPS IN THE DISTRIBUTION OF ASTEROIDS

by Yusuke Hagihara

SMITHSONIAN INSTITUTION

Washington, D.C.

1961

Publications of the Astrophysical Observatory

This series, *Smithsonian Contributions to Astrophysics*, was inaugurated in 1956 to provide a proper communication for the results of research conducted at the Astrophysical Observatory of the Smithsonian Institution. Its purpose is the "increase and diffusion of knowledge" in the field of astrophysics, with particular emphasis on problems of the sun, the earth, and the solar system. Its pages are open to a limited number of papers by other investigators with whom we have common interests.

Another series is *Annals of the Astrophysical Observatory*. It was started in 1900 by the Observatory's first director, Samuel P. Langley, and has been published about every 10 years since that date. These quarto volumes, some of which are still available, record the history of the Observatory's researches and activities.

Many technical papers and volumes emanating from the Astrophysical Observatory have appeared in the *Smithsonian Miscellaneous Collections*. Among these are *Smithsonian Physical Tables*, *Smithsonian Meteorological Tables*, and *World Weather Records*.

Additional information concerning these publications may be secured from the Editorial and Publications Division, Smithsonian Institution, Washington, D.C.

FRED L. WHIPPLE, *Director*,
Astrophysical Observatory,
Smithsonian Institution.

Cambridge, Mass.

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington 25, D.C. - Price 20 cents

Gaps in the Distribution of Asteroids

By Yusuke Hagihara [1]

When we plot the mean motions of asteroids, we observe several gaps around the values commensurable with the mean motion of Jupiter (Kirkwood, 1867), and an apparent clustering at values that are not exactly commensurable. These observations suggest that when an asteroid acquires a mean motion exactly commensurable with that of Jupiter, its motion becomes unstable and the asteroid quickly shifts into a nearby, noncommensurable orbit.

Let us consider a system consisting of an asteroid and Jupiter, both moving around the sun. According to the usual theory of perturbations, we determine the motion by the method of the variation of the elements. When we perform a first integration of the equations of motion, neglecting the perturbation due to Jupiter, we obtain an orbit that is a Keplerian ellipse with six constants of integration, called the elements of the asteroid. When we take the perturbation of Jupiter into account, the elements are no longer constant and are determined by a system of differential equations expressing the variation of the elements E as functions of the perturbation, in the form,

$$\frac{dE}{dt}=\sum_{i,j} B_{ij}\cos\,(i\lambda+j\lambda'+D_{ij}), \qquad (1)$$

$$\lambda=nt+\epsilon, \qquad \lambda'=n't+\epsilon'.$$

Here n is the mean motion, ϵ the mean longitude at the epoch, D_{ij} a function of the elements, B_{ij} also a function of the elements, i and j are integers, positive, negative, or zero, and the letters with primes represent the corresponding quantities for Jupiter. The series is arranged in an integral power series of small quantities such as the eccentricities of the orbits of the asteroid and Jupiter and the inclination of the asteroidal orbit to that of Jupiter. In the first approximation we usually integrate by neglecting the variation of the elements on the right-hand side of such equations, because the right-hand side is multiplied by the mass of Jupiter, which we consider as a small quantity of the first order. By such a formal integration, t being the only variable, we obtain the equation,

$$E=E_0+\sum_{i,j}\frac{B_{ij}}{in+jn'}\,(\sin\,i\lambda+j\lambda'+D_{ij}). \qquad (2)$$

In this way the solution is expressed in the form of a trigonometric series in which each term represents a periodic inequality with the period $2\pi/(in+jn')$. Thus the variation of the elements is expressed formally as a sum of periodic terms and the orbit is considered to be stable in a certain sense.

However, if $i=j=0$, then the corresponding term is expressed as a linear function of t and the term is called secular. It can be shown from Poisson's theorem that there is no secular term in the expression for the semimajor axes in the first-order perturbation. If $in+jn'=0$ while $i\neq j\neq 0$, then the corresponding term is called critical. If $in+jn'\neq 0$ but is very small, then the corresponding term has a small divisor $in+jn'$ in the denominator, and becomes very large compared with the remaining terms. This circumstance would spoil the formal convergence of the series obtained by formal integration of the differential equations. Actually such terms (called the "great inequalities") exist, corresponding to a small divisor of the form $2n-5n'$ in the mutual perturbation of Jupiter and Saturn, and to a smaller divisor $n-2n'$ for Uranus and Neptune. In such cases the usual theory of perturbation fails and a different method, such as that of Hansen (1857) or of Newcomb (1874, 1891), is employed for the actual computation of the perturbation.

[1] Smithsonian Astrophysical Observatory.

For the motion of an asteroid with very small values of $in+jn'$ an entirely new theory should be worked out, such as that of Glydén (1893), of Delaunay (1860, 1867), or a periodic solution due to Poincaré (1890, 1892).

It was once thought that the existence of a small divisor or the close commensurability of the mean motions would endanger the stability of the motion, and that the gaps in the distribution of the mean motions of asteroids must result from the presence of such small divisors. It should be emphasized that the series obtained in this manner is only a formal solution and that it does not by any means represent the true solution of the equations because it is not uniformly convergent, as was proved by Poincaré (1890, 1893). The problem of stability is closely related to the question of the form of the solution, which is written in its uniformly convergent form. Hence, with the ordinary formal expansion used in the perturbation theory, we cannot draw any definite conclusion as to the behavior of the motion over a comparatively long interval of time. Thus we cannot attempt to explain the occurrence of the gaps in question on the basis of such a nonconvergent series. As a matter of fact, the observations show (see fig. 1) that asteroids actually do exist with mean motions nearly commensurable with that of Jupiter, in the ratios 3 to 2 for the Hilda group, 4 to 3 for the Thule group, and 1 to 1 for the Trojan group. On the other hand, no asteroids have been found with mean motions close to the commensurable values in the ratios 2 to 1 for the Hecuba group, 3 to 1 for the Hestia group, 5 to 2, and 7 to 3. Hence we cannot accept the explanation that the gaps result from the presence of small divisors.

Hirayama (1918, 1928; see also Hagihara, 1928a, 1928b, 1928c, 1928d, 1928e) once postulated the existence of a resisting medium to explain the gaps, but later withdrew the suggestion. The families of asteroids that Hirayama (1922, 1927, 1933) discovered, which Brouwer (1950, 1951) has recently enlarged, have no direct relation to the gaps.

Similar gaps seem to exist in the rings of Saturn, as shown in figure 2. It might be thought that the occurrence of Cassini's division, the so-called Encke's division, and other smaller irregularities in the brightness distribution over the ring is related to the fact that the particles constituting the ring around Saturn have mean motions commensurable with that of Mimas or possibly of Enceladus. In addition to Cassini's division corresponding to the ratio 2 to 1 of the mean motion of a particle with that of Mimas, and Encke's division corresponding to the ratio 5 to 3 of the mean motions, Lowell (1915) reported several gaps in Saturn's rings corresponding to 3 to 1, 8 to 3, 5 to 2, 7 to 3, 9 to 4, 11 to 5 of Mimas, and to 4 to 1, 5 to 2, 7 to 3 of Enceladus. However, recent observations by Lyot at the Pic du Midi Observatory, and by others with big telescopes in the United States (Kuiper, 1960, and personal communication), show that some of the reported gaps, such as those corresponding to 5 to 2, 9 to 4 and 11 to 5 of Mimas, do not exist, and that even Encke's "division" is only an irregularity in the distribution of brightness.

Brown (1928) believed that the mean motion of an asteroid changes quickly if it once becomes nearly commensurable with that of Jupiter, and that it makes a libration about the exact value, like the pendulum near its stable equilibrium position. Thus the mean time of sojourn at the nearly commensurable value is much shorter, statistically, than that spent at noncommensurable values. Brown (1932) has therefore studied in detail the resonance phenomena in simplified dynamical systems such as a coupled pair of pendulums, but has not applied the results to the actual motion of the asteroids. This theory does not, however, explain the cases of the Thule, Hilda and Trojan groups. It has recently been proved by Roy and Ovendon (1954, 1955), from the theory of probability, that the existence of the gaps is not due to chance.

Wilkens (1933) discussed the case of multiple commensurability among the mean motions of an asteroid, of Jupiter, and of Saturn (Okyay, 1935; Urban, 1935). But according to Hagihara's (1928a, 1928b, 1928c, 1928d, 1930, 1940) study of the secular perturbation of higher degrees, supplemented by the work of Kozai (1954), this multiple commensurability has little to do with the stability of the motion, at least in the formal aspect of the problem as treated in the ordinary theory of perturbations.

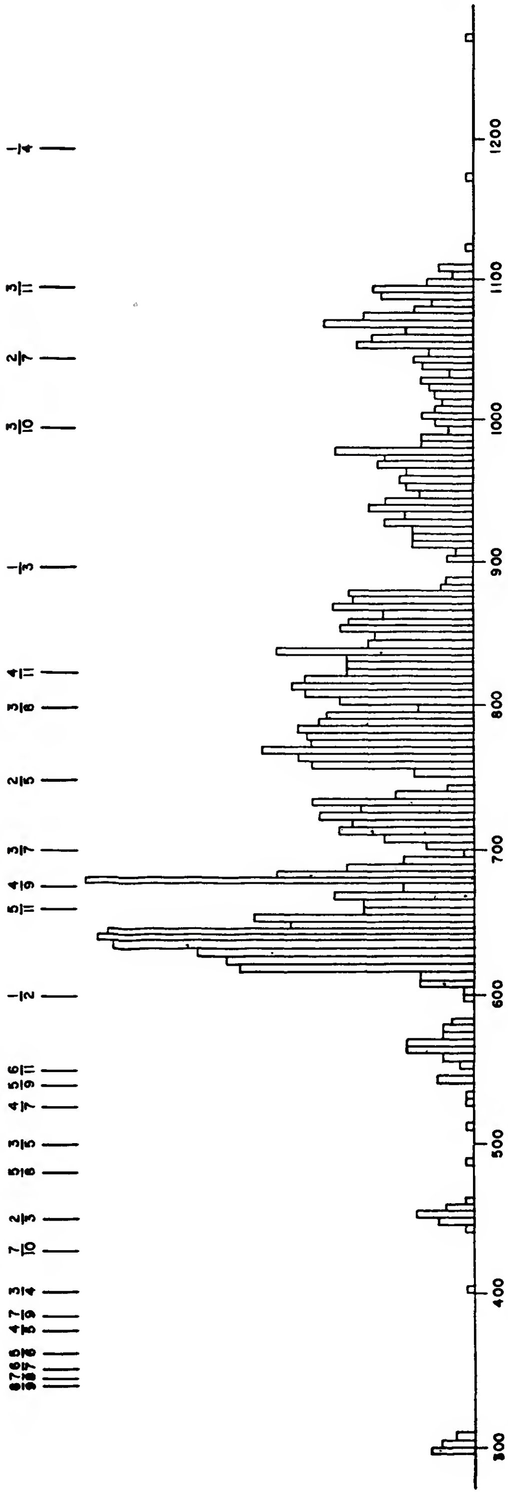

FIGURE 1.—Distribution of the mean motion of asteroids.

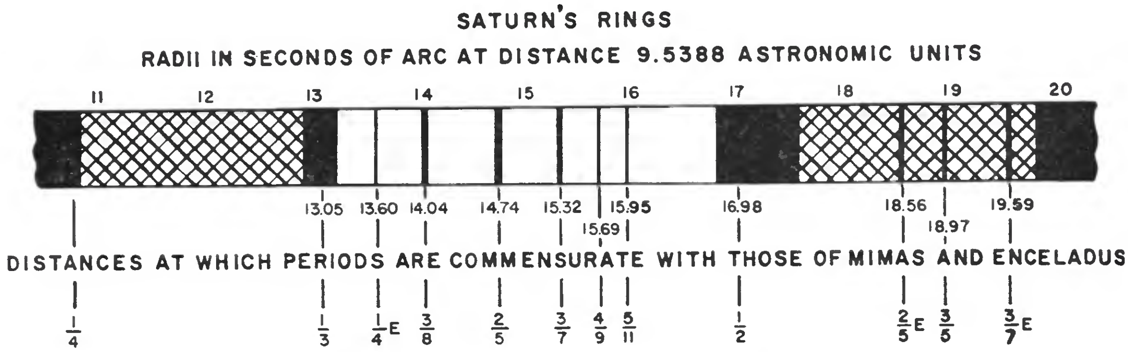

FIGURE 2.—Saturn's rings according to Lowell (1915).

While some of the asteroids apparently tend to avoid such commensurable values for their mean motions, several satellites, for example the asteroids of the Trojan, the Hilda, and the Thule groups, have mean motions that are mutually in almost exactly commensurable ratios. The first three Galilean satellites of Jupiter have for their daily mean motions the values $n_1=203^{\circ}\!.48895528$, $n_2=101^{\circ}\!.37472396$, $n_3=50^{\circ}\!.31760833$, so that $n_1-2n_2=0^{\circ}\!.73950736$, $n_2-2n_3=0^{\circ}\!.73950730$, and, even more remarkable, $n_1-3n_2+2n_3=0^{\circ}\!.00000006$ (de Sitter, 1908a, 1908b, 1909, 1918, 1928). In Saturn's system the mean motions of Mimas and Tethys are in the ratio 2 to 1, those of Enceladus and Dione in the ratio 2 to 1, and those of Titan and Hyperion 4 to 3.

The question naturally arises whether these gaps in the distribution of the mean motions of asteroids or of satellites can be reasonably explained by purely gravitational arguments. The recent cosmogonical theory of Alfvén (1954), explaining the gaps in asteroidal distribution and in Saturn's system on the basis of magnetohydrodynamical theory, is still far from being convincing evidence for or against the occurrence of the gaps (see also Kuiper, 1950, 1953a, 1953b, 1956a, 1956b, 1956c, 1956d, 1957; Rabe, 1954, 1956, 1957; Kerr and Whipple, 1951, 1954). In any case, we must first find the answer on purely gravitational grounds, before we consider the cause of the gaps on a cosmogonical basis.

Saturn has several pairs of satellites whose mean motions are commensurable. For such a pair, the configuration of the perisaturniums is such that their motions are stable. For Titan and Hyperion the conjunction occurs always at the aposaturnium of Hyperion (Newcomb, 1891; Woltjer, 1918, 1928; Brouwer, 1924). For Enceladus and Dione the conjunction occurs at the perisaturnium of Enceladus. For Mimas and Tethys the conjunction occurs at the node instead of at the perisaturnium. The conjunction of the two satellites occurs near the midpoint between their two ascending nodes on Saturn's equator. The actual configurations of these pairs of satellites are known to be stable.

It might be questioned whether the stability of the motion is affected when we take into account other disturbing bodies, such as the other satellites and the sun, and the effect of the figure of the planet. Usually such effects are not considered when the stability of the motion is discussed for the three-body problem (Brown, 1901). For the Trojan asteroids, Brown (1923, 1925) has proved that the action of the planets other than Jupiter never increases the amplitudes of the long-period libration terms without limit. Hagihara (1927) has discussed the stability of a satellite system composed of several satellites, two of which have nearly commensurable mean motions, by considering the effects of the solar perturbation and the figure of the planet. Starting with a set of intermediary nonperiodic orbits with moving pericenters for the pair of satellites, he discussed the differential equations for the disturbed motion on the basis of Poincaré's (1890, 1892) theory of characteristic exponents, and proves that the motion is stable if the positions of the pericenters of the two satellites are suitably chosen.

The most interesting case is that of the three Galilean satellites of Jupiter, I, II, and III. De Sitter (1908a, 1908b, 1909, 1918, 1928) has adopted as the intermediary orbits for the three satellites a periodic solution in which all the three satellites have the same perijove motion. He also proved the stability of this solution by Poincaré's method of characteristic exponents. The perijove of the second satellite lies opposite those of the first and third satellites. When the first and third satellites are at a certain epoch in conjunction, and the second satellite is in opposition, all in their apojoves, then the motion is unstable. But when the first and third satellites are in conjunction and the second satellite is in opposition, all in their perijoves at a certain epoch, then the motion is stable. This case of stable motions is what we actually observe in nature.

The situation is entirely different for the gaps in Saturn's rings and in the asteroidal distribution. Goldsbrough (1921, 1922, 1924) has discussed the stability of the motion of small bodies with mean motions commensurable with that of Mimas (Pendse, 1933, 1935, 1937). He studied the motion of small bodies situated at the vertices of an equilateral polygon having Saturn at its center and rotating around Saturn. This configuration is what was called a central figure by Dziobek (1900; see also Andoyer, 1906; Meyer, 1933; Hölder, 1929). Goldsbrough considered a small deviation of the bodies from this equilibrium position and discussed the solution, from the point of view of stability, of the differential equations of the motion when disturbed by Mimas; he proved the instability of the motion around the ratios of the mean motions in the principal commensurable values corresponding to Cassini's and Encke's divisions. Hagihara (1940) has studied the system of differential equations for the disturbed motion with parametric representation, employing Peano-Baker's matrix method (Baker, 1916). From the theory of nonlinear integrodifferential equations, Lichtenstein (1923, 1924, 1933, 1932, 1931) has proved the existence of periodic solutions and found that the deviation from the equilibrium position propagated in a wave form.

The motion of a "characteristic" asteroid, having a mean motion nearly commensurable with that of Jupiter, has been studied theoretically on the basis of the corresponding periodic solution (Hill, 1902a, 1902b; Poincaré, 1890, 1892, 1902a, 1902b; Andoyer, 1903; Schwarzschild, 1903; Wilkens, 1913; Heinrich, 1912, 1922, 1925, and others). Poincaré (1890, 1892) has proved the existence of periodic solutions for a continuous set of values of a parameter μ in the neighborhood of $\mu=0$ for the case in which a generating periodic solution exists for $\mu=0$. Consider two planets of small mass, revolving around the Sun. Suppose that the generating periodic solution for the motion of the two planets is such that their orbits around the Sun have nonzero eccentricities but perihelia with a special configuration of position, mean motions that are commensurable, a ratio of eccentricities that is related to the ratio of the semimajor axes; suppose further that the nearby periodic solutions are such that their perihelia are immovable; then a continuous set of periodic solutions exists with a continuous set of values of μ differing slightly from zero, which Poincaré called periodic solutions of the second sort.

Another approach to the character of the motion, when there is a small divisor $in+jn'$, considers the corresponding periodic term with a long-periodic variation called libration. Brown (1911a, 1911b, 1911c, 1912; Brown and Shook, 1933) has studied the libration in asteroidal motion by his own ingenious method and Hagihara (1944) has based his discussion on Delaunay's method used in the theory of perturbation. In view of the approximate character of such analytical theories, Hirayama (1918, 1928; Wilkens, 1927, 1930; Hirayama and Akiyama, 1937a, 1937b; Ura and Takenouchi, 1951; Kozai, 1952; 1953) undertook the computation of the libration by the method of special perturbation and verified the analytical theory of Brown (1912).

These methods have shown that the presence of small divisors in the ordinary perturbation theory does not make the situation worse as to the stability of the motion. On the contrary, the commensurability of the mean motions strengthens the durability and the tendency to keep the existing configuration, provided that the perihelia are suitably located to produce a stable configuration.

We may now summarize the main conclusion of the present discussion. The stability of a satellite system is not weakened but strengthened by the presence of commensurable mean motions, provided that the relative configuration is the one which is stable. But, if the system contains many members, and the positions of the pericenters and the longitudes of the satellites in their orbits are arbitrary, as for Saturn's rings and, presumably, the asteroidal rings, then gaps could occur in the distribution of the mean motions, because of the accumulated effects of disturbing actions by the neighboring small masses passing close by. In fact, there are very few members in the Hilda group of asteroids and only one member in the Thule group. Until a detailed precise theory is worked out on the motion of such characteristic asteroids, the present point of view (Hagihara, 1957) may serve as a tentative theoretical explanation of the occurrence of gaps in the distribution of the mean motions of asteroids and the gaps in Saturn's rings. We await new evidence such as that to be provided by the photometric study of asteroids by Kuiper and his colleagues, and the infrared study of Saturn's rings by Whipple and his colleagues, which will help in clearing up this difficult problem.

References

Alfvén, H.
1954. On the origin of the solar system. 194 pp.

Andoyer, H.
1903. Contribution à la théorie des petites planètes dont le moyen mouvement est sensiblement double de celui de Jupiter. Bull. Astron., vol. 20, pp. 321–356.
1906. Sur l'équilibre relatif de n corps. Bull. Astron., vol. 23, pp. 50–59.

Baker, H. F.
1916. On certain linear differential equations of astronomical interest. Phil. Trans Roy. Soc. London, ser. A, vol. 216, pp. 129–186.

Brouwer, D.
1924. A new determination of the mass of Titan by Hill's method. Bull. Astron. Inst. Netherlands, vol. 2, pp. 119–120.
1950. Families of minor planets and related distributional problems. Astron. Journ., vol. 55, pp. 162–163.
1951. Secular variations of the orbital elements of minor planets. Astron. Journ., vol. 56, pp. 9–32.

Brown, E. W.
1901. Modern methods of treating dynamical problems and in particular the problem of three bodies. Bull. Amer. Math. Soc., vol. 8, pp. 103–113.
1911a. On a new family of periodic orbits in the problem of three bodies. Monthly Notices Roy. Astron. Soc. London, vol. 71, pp. 438–454.
1911b. On the oscillating orbits about the triangular equilibrium points in the problem of three bodies. Monthly Notices Roy. Astron. Soc., London, vol. 71, pp. 492–502.
1911c. Errata for "On a new family of periodic orbits in the problem of three bodies." Monthly Notices Roy. Astron. Soc. London, vol. 71, p. 716.
1912. On librations in planetary and satellite systems. Monthly Notices Roy. Astron. Soc. London, vol. 72, pp. 609–630.
1923. Theory of the Trojan group of asteroids. Trans. Astron. Obs. Yale Univ., vol. 3, pp. 1–47.
1925. Theory of the Trojan group of asteroids. (Continued.) Trans. Astron. Obs. Yale Univ., vol. 3, pp. 83–133.
1928. Resonance in the solar system. Bull. Amer. Math. Soc., vol. 34, pp. 265–289.
1932. Elements of the theory of resonance. 60 pp.

Brown, E. W., and Shook, C. A.
1933. Planetary theory. 302 pp.

Delaunay, C. E.
1860. Theorie du mouvement de la lune. I. Mem. Acad. Sci. Inst. Imp. France, vol. 28, 883 pp.
1867. Theorie du mouvement de la lune. II. Mem. Acad. Sci. Inst. Imp. France, vol. 29, 931 pp.

Dziobek, O.
1900. Ueber einen merkwürdigen Fall des Vielkörperproblems. Astron. Nachr., vol. 152, pp. 33–46.

Glydén, H.
1893. Traité analytique des orbites absolues des huit planètes principales. 573 pp.

Goldsbrough, G. R.
1921. The influence of satellites upon the form of Saturn's ring. Phil. Trans. Roy. Soc. London, ser. A, vol 222, pp. 101–130.
1922. The cause of Encke's division in Saturn's ring. Proc. Roy. Soc. London, ser. A, vol. 101, pp. 280–289.
1924. On the possible ellipticity of Saturn's ring. Proc. Roy. Soc. London, ser. A, vol. 106, pp. 526–552.

Hagihara, Y.
1927. A preliminary study on the stability of a certain satellites-system. Japanese Journ. Astron. Geophys., vol. 5, pp. 1–7 9.

1928a. On the theory of the secular perturbation of higher degrees. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 10, pp. 1–14.
1928b. On the theory of the secular perturbation of higher degrees. II. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 10, pp. 34–59.
1928c. On the theory of the secular perturbation of higher degrees. III. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 10, pp. 87–113.
1928d. On the theory of the secular perturbation of higher degrees. IV. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 10, pp. 127–156.
1928e. Einige Bemerkungen über die Hecubalücke in der Verteilung der kleinen Planeten. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 10, pp. 15–24.
1930. On the theory of the secular perturbation of higher degrees. A correction. Proc. Phys.-Math. Soc. Japan, ser. 3, vol. 12, pp. 22–23.
1940. Application of matrix method to integration problems. *In* Festschrift für Elis Strömgren, pp. 58–81.
1944. On the general theory of libration. Japanese Journ. Astron. Geophys., vol. 21, pp. 29–43.
1957. Stability in celestial mechanics. Tokyo, 106 pp.

Hansen, P. A.
1857. Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten. Mitg. Kön. Sächs. Ges. Wiss. Leipzig, vol. 5, p. 41ff.

Heinrich, W. W.
1912. Über die periodischen Bahnen vom Typhus $(p+2)/p$, insbesondere über den Fall 5/3. Astron. Nachr., vol. 192, pp. 325–336.
1922. Sur certaines classes nouvelles de solutions de bifurcation à période séculaire du problème astéroïdique des trois corps. Bull. Astron., ser. 2, vol. 2, pp. 425–458.
1925. Nouvelles classes de solutions séculaires du problème astéroïdique de trois corps. Monthly Notices Roy. Astron. Soc. London, vol. 85, pp. 625–639.

Hill, G. W.
1902a. Illustrations of periodic solutions in the problem of three bodies. I. Astron. Journ., vol. 22, pp. 93–97.
1902b. Illustrations of periodic solutions in the problem of three bodies. II. Astron. Journ., vol. 22, pp. 117–121.

Hirayama, K.
1918. Researches on the distribution of the mean motions of asteroids. Journ. Coll. Sci. Imp. Univ. Tokyo, vol. 41, No. 3, 58 pp.
1922. Families of asteroids. Japanese Journ. Astron. Geophys., vol. 1, pp. 55–93.
1927. Families of asteroids. II. Japanese Journ. Astron. Geophys., vol. 5, pp. 137–162.
1928. Note on an explanation of the gaps of the asteroidal orbits. Astron. Journ., vol. 38, pp. 147–148.
1933. Present state of the families of asteroids. Proc. Imp. Acad. Tokyo, vol. 9, pp. 482–485.

Hirayama, K., and Akiyama, K.
1937a. On the motion of the asteroid Hilda (153). Proc. Imp. Acad. Tokyo, vol. 13, pp. 191–195.
1937b. Libration of the asteroid Hilda. Japanese Journ. Astron. Geophys., vol. 15, pp. 137–148.

Hölder, E.
1929. Mathematische Untersuchungen zur Himmelsmechanik. Math. Zeitschr., vol. 31, pp. 197–257.

Kerr, F. J., and Whipple, F. L.
1951. Possible explanations of the secular acceleration of Phobos and Jupiter V. Astron. Journ., vol. 56, p. 131. (Abstract.)
1954. On the secular accelerations of Phobos and Jupiter V. Astron. Journ., vol. 59, pp. 124–127.

Kirkwood, D.
1867. Meteoric astronomy. Ch. 13. 129 pp.

Kozai, Y.
1952. Preliminary studies on the motion of asteroid Thule (279). I. Publ. Astron. Soc. Japan, vol. 3, pp. 184–194.
1953. Preliminary studies on the motion of asteroid Thule (279). II. Publ. Astron. Soc. Japan, vol. 4, pp. 172–179.
1954. On the secular perturbations of asteroids. Publ. Astron. Soc. Japan, vol. 6, pp. 41–66.

Kuiper, G. P.
1950. On the origin of asteroids. Astron. Journ., vol. 55, p. 164.
1953a. Satellites, comets, and interplanetary material. Proc. Nat. Acad. Sci., Washington, vol. 39, pp. 1153–1158.
1953b. Note on the origin of the asteroids. Proc. Nat. Acad. Sci. Washington, vol. 39, pp. 1159–1161.
1956a. The formation of the planets. I. Journ. Roy. Astron. Soc. Canada, vol. 50, pp. 57–68.
1956b. The formation of the planets. II. Journ. Roy. Astron. Soc. Canada, vol. 50, pp. 105–121.
1956c. The formation of the planets. III. Journ. Roy. Astron. Soc. Canada, vol. 50, pp. 158–178.
1956d. The origin of the satellites and the Trojans. *In* A. Beer, ed., Vistas in astronomy, vol. 2, pp. 1631–1666.

1957. Further studies on the origin of Pluto. Astrophys. Journ., vol. 125, pp. 287–289.
In press. The solar system. Vol. 3.

LICHTENSTEIN, L.
1923. Untersuchungen über die Figur der Himmelskörper. Vierte Abhandlung. Zur Maxwellschen Theorie der Saturnringe. Math. Zeitschr., vol. 17, pp. 62–110.
1924. Untersuchungen über die Figur der Himmelskörper. Fünfte Abhandlung. Neue Beiträge zur Maxwellschen Theorie der Saturnringe. *In* Probleme der Astronomie; Festschrift für Hugo v. Seeliger, pp. 200–227.
1931. Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-Differentialgleichungen. 164 pp.
1932. Untersuchungen über die Gestalt der Himmelskörper. Sechste Abhandlung. Weitere Beiträge zur Maxwellschen Theorie der Saturnringe. Annali R. Scuola Norm. Sup. Pisa, ser. 2, vol. 1, pp. 173–213.
1933. Untersuchungen über die Gestalt der Himmelskörper. Siebente Abhandlung. Beiträge zur Maxwellschen Theorie der Saturnringe. Anfangswertproblem. Math. Zeitschr., vol. 37, pp. 424–445.

LOWELL, P.
1915. Memoir on Saturn's rings. Mem. Lowell Obs., vol. 1, No. 2, 22 pp.

MEYER, G.
1933. Solutions voisines des solutions de Lagrange dans le problème des *n* corps. Ann. Obs. Bordeaux, vol. 17, pp. 77–252.

NEWCOMB, S.
1874. On the general integrals of planetary motion. Smithsonian Contr. Knowl., No. 281, 31 pp.
1891. On the motion of Hyperion. A new case in celestial mechanics. Astron. Papers Amer. Ephemeris and Nautical Almanac, Vol. 3, pp. 345–371.

OKYAY, T.
1935. Über die mehrfachen Kommensurabilitäten im System Planetoid-Jupiter-Saturn. Astron. Nachr., vol. 255, pp. 277–300.

PENDSE, C. G.
1933. On the stability of Saturn's rings. Philos. Mag., ser. 7, vol. 16, pp. 575–580.
1935. The theory of Saturn's rings. Philos. Trans. Roy. Soc. London, ser. A, vol. 234, pp. 145–176.
1937. Solution of a set of differential equations in the theory of Saturn's rings. Philos. Mag., ser. 7, vol. 23, pp. 425–440.

POINCARÉ, H.
1890. Sur le problème des trois corps et les équations de la dynamique. Acta Math., vol. 13, pp. 1–270.
1892. Méthodes nouvelles de la mécanique céleste. vol. 1, 385 pp.
1893. Les méthodes nouvelles de la mécanique céleste. Vol. 2, 480 pp.
1902a. Sur les planètes du type d'Hécube. Bull. Astron., vol. 19, pp. 289–310.
1902b. Les solutions périodique et les planètes du type d'Hécube. Bull. Astron., vol. 19, pp. 177–198.

RABE, E. K.
1954. Escaping satellites of Jupiter. Astron. Journ., vol. 59, p. 190. (Abstract.)
1956. On the origin of the Kirkwood gaps and the minor planet families. Zeitschr. Astrophys., vol. 40, pp. 107–119.
1957. On the origin of Pluto and the masses of the protoplanets. Astrophys. Journ., vol. 125, pp. 290–295.

ROY, A. E., AND OVENDON, M. W.
1954. On the occurrence of commensurable mean motions in the solar system. Monthly Notices Roy. Astron. Soc. London, vol. 114, pp. 232–241.
1955. On the occurrence of commensurable mean motions in the solar system. II. The mirror theorem. Monthly Notices Roy. Astron. Soc. London vol. 115, pp. 296–309.

SCHWARZSCHILD, K.
1903. Über die periodischen Bahnen vom Hecubatypus. Astron. Nachr., vol. 160, pp. 385–400.

SITTER, W. de
1908a. On the masses and elements of Jupiter's satellites and the mass of the system. Proc. Kon. Akad. Wetenschappen Amsterdam, vol. 10, pp. 653–673.
1908b. On the masses and elements of Jupiter's satellites and the mass of the system. II. Proc. Kon. Akad. Wetenschappen Amsterdam, vol. 10, pp. 710–729.
1909. On the periodic solutions of a special case of the problem of four bodies. Proc. Kon. Akad. Wetenschappen Amsterdam, vol. 11, pp. 682–715.
1918. Outlines of a new mathematical theory of Jupiter's satellites. Ann. Sterrewacht Leiden, vol. 12, pp. 1–55.
1928. Orbital elements determining the longitudes of Jupiter's satellites, derived from observations. Ann. Sterrewacht Leiden, vol. 16, pp. 1–94.

URA, T., AND TAKENOUCHI, T.
1951. On special perturbations by the variations of elements. Publ. Astron. Soc. Japan, vol. 2, pp. 93–101.

URBAN, K.
1935. Allgemeine Jupiterstörungen des Planetoiden 202 Chryseïs. Mit besonderer Berücksichtingung der doppelten Kommensurabilität zu Jupiter und Saturn. Astron. Nachr., vol. 256, pp. 109–154.

WILKENS, A.

1913. Über die praktische Verwendung der Poincaréschen periodischen Bahnen im Sonnensystem. Astron. Nachr., vol. 195, pp. 385–412.

1927. Zur Erklärung der Planetoidenlücken im Sonnensystem. Sitz. Bay. Akad. Wiss. München, No. 2, pp. 197–204.

1930. Über einen speziellen Fall des Problems der drei Körper. Astron. Nachr., vol. 240, pp. 201–232.

1933. Über das Problem der mehrfachen Kommensurabilitäten im Sonnensystem. Sitz. Bay. Akad. Wiss. München, No. 1, pp. 71–101.

WOLTJER, J., JR.

1918. Investigations in the theory of Hyperion. Dissertation. Leiden.

1928. The motion of Hyperion. Ann. Sterrewacht Leiden, vol. 16, pt. 3, 139 pp.

U.S. GOVERNMENT PRINTING OFFICE: 1961