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THIS WORK

"THE CORRECTION OF SEXTANTS for Errors of Eccentricity and Graduation."

BY

JOSEPH A. ROGERS,

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THE CORRECTION OF SEXTANTS

FOR ERRORS OF

ECCENTRICITY AND GRADUATION.

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JOSEPH A. ROGERS.

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ADVERTISEMENT.

Improvement in the construction of instruments of precision is necessarily preceded by the development of means for detecting the errors of those already in use. The sextant, though primarily an instrument for the traveller, will, when carefully made and handled, give results of a remarkable degree of precision ; and it is a matter of great importance that even for the ordinary purposes of navigation every precaution should be taken to free it from all classes of instrumental error. Inaccuracies of mechanical construction, though very minute from the artisan's standard, are greater in effect than even experienced navigators sometimes realize.

An apparatus for investigating and determining such errors of the sextant, devised by Mr. Rogers (the theory and use of which is described in the present treatise), has been extensively employed by the officers of the United States Navy in testing sextants before their issue to the ships of the naval service.

It may be added that, in the constant re-action between advances in science and art, the reciprocal benefits of which should always be freely acknowledged, the skill of the instrument-maker (especially in the accuracy of graduation of circular measures) is still behind the delicate requirements of modern investigation ; and it is confidently believed that methods of detecting minute inaccuracies will result, as in the past, in stimulating the artisan to further refinements in instrumental construction.

S. P. Langley,

Secretary of Smithsonian Institution.

Washington, *December*, 1890.

THE CORRECTION OF SEXTANTS FOR ERRORS

OF

ECCENTRICITY AND GRADUATION.

BY JOSEPH A. ROGERS.

The sextant of reflection, in consequence of its portability and the ease and rapidity whith which it affords results of considerable accuracy in the hands of a capable observer, is so serviceable in navigation, surveying, and in determining latitude and time for astronomical purposes, that no reasonable proposition to increase the precision of observations made with an instrument of such general usefulness can be devoid of interest. One of the most obvious sources of avoidable error is a non-coincidence of the axis of rotation of the index-bar with the axis of the graduated arc. The necessary lightness of all the parts also entails a certain liability to irregularities in the rotating motion, due to flexure of the slender axis and its connections, or to wabbling when the axis is not properly supported at both ends. With good workmanship, and proper attention to the condition of the instrument when used, these irregularities may be kept within narrow limits, but there is reason to believe that they are sometimes greater than is commonly suspected. When the arc is extended into a complete circle with opposite indices, the effect of eccentricity is eliminated, and that of irregular rotation is much diminished, if not entirely compensated. These advantages of the circle have been frequently urged, and are indisputable, but they cannot be secured without some sacrifice of other desirable qualities. An instrument which is designed to be supported in the hand when used, must be light and compact; with equal bulk and weight the limb of the sextant has a greater radius than that of the circle, and consequently permits a closer reading of the angles measured, while the uncompensated effects of irregular rotation in the former may be lessened by a somewhat more massive and substantial construction of the axis and parts supporting it; moreover, the sextant is the less costly of the two, and is, perhaps, in shape, rather better adapted to convenience in manipulation. The settlement of these conflicting claims by experience has been upon the whole unfavorable to the circle, for its employment is now quite exceptional, and there is apparently no reason to anticipate a reversal of this verdict. It is very desirable, therefore; to possess some efficient means of correcting the errors which are peculiar to the sextant.

The eccentric corrections of a sextant may be deduced from measurements made with it of three or more known angular distances. A trustworthy determination requires several angles, of such magnitudes that no part of the arc shall be very far from one of the readings obtained. The known angles may be the apparent distances between stars, computed from the positions given in the catalogues, or the angles subtended by well-marked terrestrial objects, measured by a theodolite or otherwise. One of these methods will usually suffice for an observer who wishes to obtain corrections for his own instrument, but neither of them can be regarded as generally available when large numbers of sextants are to be examined, for besides the laborious computation involved, stars are seen only at night and in clear weather, while a spot where suitable terrestrial objects at proper angles and sufficient distances are always visible is sometimes difficult to find.

During the latter part of the civil war the writer, then an assistant in the U. S. Naval Observatory, was required to superintend the repairs of sextants returned in unserviceable condition from the various fleets. Some of these instruments bore marks of violence, and it was felt to be extremely desirable that none of them should be reissued until their integrity had been verified by some adequate test. A plan for making such tests was accordingly formed, but circumstances did not permit it to be carried out at that time. Not long afterward, however, while engaged in a similar occupation at the Hydrographic Office, the apparatus about to be described was constructed. In the meantime a description was published* of Mr. T. Cooke's apparatus, which had recently been set up at the Kew Observatory for the same purpose. Disregarding minor details, the apparatus at Kew consists essentially of a series of horizontal collimators equidistant from, and directed toward, a central point where the sextant under examination is supported in a horizontal position by a small table. The collimators are firmly secured to a curved pier of substantial masonry; the angles which their axes make with each other are measured by a theodolite temporarily placed on the central table for that purpose, and are presumed to remain nearly constant. A remeasurement of these angles with any sextant affords a comparison between its graduation and that of the theodolite, at points depending upon the number and disposition of the collimators, which at Kew indicate five axial directions, including angles approximating 15°, 30°, 45°, 60°, 75°, 90°, 105°, and 120°.

* Balfour Stewart, Proc. Roy. Soc., London, 1868, xvi, 2.

The apparatus constructed at the Hydrographic Office is represented by Fig. 1, in which A is a stout walnut plank $4\frac{1}{2}$ feet long, 2 feet wide, and nearly 2 inches thick, supporting the other parts, and mounted upon legs at a convenient height above the floor. The horizontal collimator B, having an aperture of 2 inches, and a focal length of 19 inches, is firmly secured to the base-board at an elevation nearly equal to that of the telescope of a sextant in the position it occupies while undergoing examination. In the focal plane is a thin metallic plate with a minute circular hole drilled through it, which appears as a sharply defined bright disc a little less than $2'$ in diameter, when illuminated from behind and

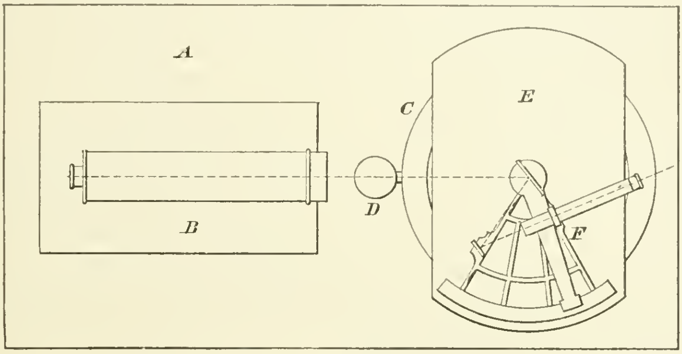

Fig. 1.

viewed with a telescope. At C is a graduated circle revolving upon a vertical axis, and supported by a tripod base after the fashion of a theodolite. One foot of the tripod, resting on its foot-screw, is directed toward the collimator as shown at D. The axis of the circle must be kept perpendicular to the line of collimation; this adjustment is made by means of the foot-screw D, and has usually been tested by placing an alidade with plane sights on the face of the circle, as extreme precision is not essential; but a reversible collimator, having collars of equal size supported by a stand laid on the table, would be a better device. The circle is divided to $5'$, and reads to $3''$ by four equidistant verniers; its diameter is about $11\frac{1}{2}$ inches. Three arms radiate from the top of the vertical axis and bear leveling screws at their extremities supporting the brass table E, which has adjustable clamps upon its upper surface to receive the legs of a sextant, and prevent them from moving in any direction. A small circular mark in the center of the table indicates the location of the vertical axis. A sextant, F, laid upon the table, with the axes of the graduated arc and circle approximately coincident, can be secured in that position by the clamps, and the plane of the arc can then be brought parallel to that of the circle by means of the table-screws. The arc now rotates in its own plane when the table revolves, and the extent of any such motion can be accurately measured by the verniers of the circle. The index-bar being in any position desired, with the direct view through the horizon-glass cut off by interposing the colored screens, let the table be turned until the reflected image of the collimator-mark is bisected by one of the telescope wires, and let both sextant and circle be read. Now move the index-bar into any other position, bring the mark back upon the same wire by turning the table in the opposite direction, and again read the sextant and circle. In this process the angle of rotation has obviously been measured by both instruments, and the difference between the two results is, therefore, the correction of the sextant for that angle, if the readings of the circle have been duly corrected for their own errors. The circle referred to has never been thoroughly investigated, but a preliminary examination showed that its errors are probably small; a double axis permits any part of the graduation to be used, and the effect of eccentricity is, of course, always eliminated. Except the circle and its immediate appendages, which were taken from a theodolite of exquisite workmanship by Gambey, of Paris, this apparatus, intended rather to illustrate a proposed system of examination than for service, was roughly made, but it proved to be efficient, and was sent to the Naval Observatory some years afterward, where it is, I believe, still occasionally used.

As compared with Mr. Cooke's system of collimators the apparatus just described presents some important advantages. Direct reference to the circle which serves as a standard is preferable to a comparison depending upon the permanence of intermediate arrangements, even such as are believed to be of a stable character. Instead of being restricted to a few angles, which it must be inconvenient, at least, to change, any part of the graduation may be tested at pleasure; the position of every line on the arc can be verified if that is desired; and the portability of an apparatus which can be moved from one room to another by a couple of persons is a minor point in its favor. The collimators, on the other hand, conveniently dispense with all readings except those of the sextant during the examination.

Before commencing the examination of a sextant, the principal adjustments must be tested, and rectified if necessary; both mirrors must be truly parallel to the axis, and the line of collimation parallel to the plane of the limb; the state of the index correction is immaterial. The telescope of the sextant may be used when of sufficient power, but can advantageously be replaced by a special one, magnifying at least ten diameters, with a fine wire in the middle of the field crossed exactly at right angles by two other wires which inclose a central space of convenient width.

These conditions having been satisfactorily established, the course of procedure is as follows: Lay the sextant on the table of the apparatus with the central leg, or axis guard, nearly in the middle of the small circular mark, bring the clamps in opposing directions against the three legs, and secure them with just sufficient pressure to prevent any subsequent displacement. Cover the horizon glass with all its colored screens, bring the reflected image of the collimator-mark into the field of the telescope, verify the focus, and see that the wire to be used in bisection is perpendicular to the path described by the mark when the index-bar is moved. In any position of the index-bar the mark can now be brought into the field by turning the table, and vice versa within the range of the sextant. By means of the table-screws make the axis of the sextant parallel to (and nearly coincident with) that of the circle, which will be the case when in any two, and consequently in all, positions of the index-bar the image passes through the center of the field as the table rotates. This is most readily accomplished by first turning either one of a pair of the screws which has been brought parallel to the axis of the collimator, and afterward turning the third screw when the same pair has been placed at right angles to that axis. If the index-mirror is not parallel to the axis of the sextant, the image cannot be made to pass through the center of the field in more than two positions of the index-bar, for the rotation of the axis causes that part of the ray passing through the telescope axially which lies beyond the mirror, to describe, not a plane, but a conical surface, only two elements of which can be brought parallel to the plane of the circle. It is advisable, therefore, before proceeding further, to see that the image passes through the field centrally when the index-bar is near the middle, and also when near each end, of its range.

The examination has usually been made by first setting the sextant at 0°, bisecting the collimator mark with the vertical wire by moving the tangent screw of the circle, and recording the reading of one pair of opposite verniers. After repeating this operation for each line of the graduation included in the inspection, a final reading is made at 0°, which ought to agree closely with the first one unless some displacement has intervened. With ordinary care this accident rarely happens, and if it does occur, its location should be revealed by a break in the series of readings.

The arc of a sextant, being the result of a fallible human effort to materialize a geometrical conception, must be regarded as more or less imperfect in every detail. There is everywhere, however, a tolerably close approximation to a certain supposititious graduation absolutely free from error, which may be called the mean arc, so situated that the algebraic sum of all the distances between corresponding lines of the actual and mean arcs is equal to zero. Every reading of the sextant will, therefore, require a correction consisting of two parts—one due to the eccentric position of the axis, and the other a local correction equal to the difference between the actual reading and the corresponding reading on the mean arc. In Fig. 2 let $G\ H$ represent the mean arc of a sextant of which I is the center and G the zero of graduation, the axis of the index-mirror being at K. If the index bar is in the position $K\ H$ the reading of the vernier, γ', will be twice the angle $G\ I\ H$, whereas the true reading, γ, is

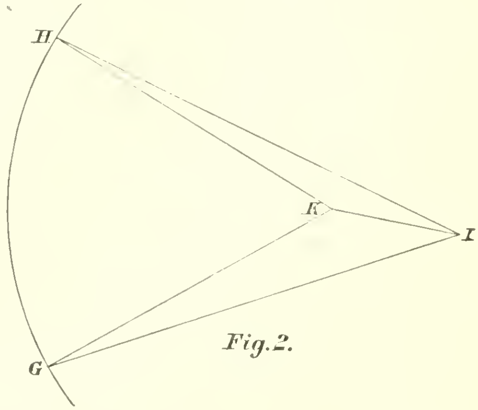

Fig. 2.

twice the angle $G\ K\ H$ by which the index is removed from 0°. Let $G\ I\ K = \varepsilon$, $I\ K = e$, and $K\ G = K\ H = L$. Now $G\ K\ H - G\ I\ H = K\ H\ I + K\ G\ I$, and the correction sought is therefore :

$$\gamma - \gamma' = 2\ K\ H\ I + 2\ K\ G\ I.$$

But $\sin K\ H\ I = \dfrac{e \sin\left(\frac{1}{2}\gamma' - \varepsilon\right)}{L} = \dfrac{e \sin \frac{1}{2}\gamma' \cos \varepsilon - e \cos \frac{1}{2}\gamma' \sin \varepsilon}{L}$, or since $K\ H\ I$ is so small as to be sensibly equal to $\dfrac{\sin K\ H\ I}{\sin 1''}$,

$$K\ H\ I = \frac{e \sin \frac{1}{2}\gamma' \cos \varepsilon - e \cos \frac{1}{2}\gamma' \sin \varepsilon}{L \sin 1''}.$$

Also, $\sin K\ G\ I = \dfrac{e \sin \varepsilon}{L}$, and

$$K\ G\ I = \frac{e \sin \varepsilon}{L \sin 1''}.$$

The substitution of these values gives :

$$\gamma - \gamma' = \frac{2\ e \cos \varepsilon}{L \sin 1''} \sin \tfrac{1}{2}\gamma' + \frac{2\ e \sin \varepsilon}{L \sin 1''}\left(1 - \cos \tfrac{1}{2}\gamma'\right),$$

and by making

$$\frac{2\ e \cos \varepsilon}{L \sin 1''} = \mathrm{A}, \qquad \frac{2\ e \sin \varepsilon}{L \sin 1''} = \mathrm{B} \tag{1}$$

the correction of any angle γ' measured from mean 0° becomes :

$$\gamma - \gamma' = A \sin \tfrac{1}{2}\gamma' + B\left(1 - \cos \tfrac{1}{2}\gamma'\right) \tag{2}$$

If we had access to the mean arc, equation (2) would enable us to find the correction for any angle whatever by making three comparisons with the standard circle, the index being set successively at 0°, and at any two other points. For the differences between the reading of the circle with the index at 0° and each of the other two readings would furnish two values of γ, in which the error of observation could be diminished by repetition to any extent desired. By substituting these values of γ and the corresponding ones of γ' in (2), two equations would be obtained determining A and B, which substituted in the same formula (2) would afford an equation giving the correction $\gamma - \gamma'$ for any value of γ'. From the definition of the mean arc it follows that the same result would be obtained from readings made with the index set successively upon every line of the actual graduation. Let R be the circle reading corresponding to any setting S of the index, and Z an assumed value of the reading when the index is at 0° of the mean arc, the true reading being $Z + X$, in which X is unknown. Then, disregarding in each case, for the reason just given, the deviation of the setting from the corresponding position on the mean arc, $\gamma' = S$, and $\gamma = R - (Z + X)$.*

Substituting these values in (2), and making

$$R - S - Z = D, \tag{3}$$

each observation will furnish an equation of the form:

$$A \sin \tfrac{1}{2} S + B(1 - \cos \tfrac{1}{2} S) + X = D. \tag{4}$$

If M is the number of observations, the normal equations will be:

$$\left.\begin{aligned} &A\,[\sin^2 \tfrac{1}{2} S] + B\,[(1 - \cos \tfrac{1}{2} S) \sin \tfrac{1}{2} S] + X[\sin \tfrac{1}{2} S] - [D \sin \tfrac{1}{2} S] = 0 \\ &A\,[\sin \tfrac{1}{2} S (1 - \cos \tfrac{1}{2} S)] + B\,[(1 - \cos \tfrac{1}{2} S)^2] + X[1 - \cos \tfrac{1}{2} S] \\ &\qquad\qquad\qquad\qquad\qquad\qquad - [D\,(1 - \cos \tfrac{1}{2} S)] = 0 \\ &A\,[\sin \tfrac{1}{2} S] + B\,[1 - \cos \tfrac{1}{2} S] + M X - [D] \qquad\qquad = 0 \end{aligned}\right\} \tag{5}$$

After substituting the numerical values of the known quantities in these equations, and finding the values of A, B, and X, the correction for eccentricity of any observed reading will be given by (2). From (4) D may be obtained for any value of S, and the difference between this computed quantity and the observed value of D, for each comparison, is the local correction of the graduation, which, however, includes an unknown, and perhaps relatively large, error of observation.

An examination like that just described, embracing every line of the graduation, and repeated until the effect of errors of observation is sufficiently diminished, would afford a complete knowledge of the condition and capabilities of the instrument. For the corrections due to the position of the axis having been obtained, the local correction for each line would

* The readings of the circle are supposed to increase as the angles indicated by the sextant increase, which is actually the case in the apparatus referred to.

be the mean of the values given by the different series of comparisons, while the probable errors of the corrections, and of observations made with the sextant in question, could be deduced from the final residuals. But such a process, or even one involving only a single reading upon every line of the graduation, is far too tedious and burdensome to be practicable. We must be content in most cases with a comparatively brief and imperfect investigation, having for its object the best result that can be derived from a moderate expenditure of time and labor. With this end in view the examination must be limited to a few points equally distributed over the arc, but sufficiently numerous to warrant the assumption that they collectively represent the mean arc accurately enough for any kind of observation in which the sextant will be employed. In this way, although the attempt to secure an exact correction for every reading is abandoned, errors which similarly affect considerable portions of the arc may be corrected, and the existence of large uncorrected errors can generally be detected. The formation and solution of the normal equations usually entail a rather laborious computation, but this can be greatly abridged by a general solution, and by other convenient devices, if the examination is always made upon a uniform system of comparisons at certain invariable distances from each other. In what follows one system of this kind is presented in detail, as a type of similar systems comprising a greater or less number of comparisons.

With reference to the nature of the service expected of them, sextants may be divided into two classes, assigning to the first class instruments used in making observations for latitude and time with the artificial horizon, measuring the principal angles of surveys, etc., and to the second class those employed in the ordinary routine of navigation, and other operations of a similar grade. All the corrections of a sextant of the first class should be determined with as much precision as the capacity of such an instrument warrants, while for those of the second class it is only necessary to insure the absence of errors exceeding certain limits. In considering this subject with reference to the wants of the naval service, it was thought to be desirable that no part of the arc should be more distant in either direction from one of the points examined than the space covered by the vernier. With the usual division of the limb to 10′, reading by the vernier to 10″, this condition requires an examination at points not more than ten degrees apart. It was decided, therefore, to make circle readings with the index set successively at 0°, 10°, 20°, etc., to and including 130°. At first a comparison was also made at 140°, when the range of the sextant extended so far, but after some experience the practice was discontinued, for the definition of the collimator-mark is frequently so much impaired by extreme obliquity of the index-mirror as to render this observation of doubtful utility. A single series of such comparisons furnishes the eccentric correction with sufficient precision for a sextant of the second class, while the residuals, containing the errors of both graduation and observation, afford a trustworthy indication of the performance to be expected of the instrument under very favorable circumstances. It was proposed to subject sextants of the first class to an examination comprising several similar series of comparisons, made with different portions of the circle, the number depending somewhat upon the circumstances of each case. These repetitions yield not only improved values of the corrections for eccentricity, but also a means of separating the local errors of graduation from the errors of observation.

The normal equations obtained by making $M = 14$, and S successively $0°$, $10°$, $20°$, etc., $130°$, in (5) are:

$$\left.\begin{array}{l} 4.72172\,A + 2.17946\,B + 7.06526\,X - [D \sin \tfrac{1}{2} S] = 0, \\ 2.17946\,A + 1.09780\,B + 2.90976\,X - [D\,(1 - \cos \tfrac{1}{2} S)] = 0, \\ 7.06526\,A + 2.90976\,B + 14\,X - [D] = 0, \end{array}\right\} \quad (6)$$

from which are deduced:

$$\left.\begin{array}{l} A = -1.5671\,[D] + 7.6467\,[D \sin \tfrac{1}{2} S] - 11.0275\,[D(1 - \cos \tfrac{1}{2} S)], \\ B = 1.8384\,[D] - 11.0275\,[D \sin \tfrac{1}{2} S] + 17.9311\,[D(1 - \cos \tfrac{1}{2} S)], \\ X = 0.4802\,[D] - 1.5671\,[D \sin \tfrac{1}{2} S] + 1.8384\,[D(1 - \cos \tfrac{1}{2} S)], \end{array}\right\} \quad (7)$$

and also:

$$X = 0.0714\,[D] - 0.5047\,A - 0.2078\,B. \quad (8)$$

From each observed value of D let the corresponding value computed by (4) be subtracted; the remainder, which may be designated $O - C$, is the sum of an observed local correction of the graduation, and an error of observation. Any single observation made under circumstances as favorable as those attending the examination, and corrected for eccentricity only, will, therefore, contain an error the most probable estimate of which is:

$$0.6745 \sqrt{\frac{[(O - C)^2]}{14 - 3}} = \sqrt{0.04136\,[(O - C)^2]}, \quad (9)$$

if the supposition is made that the errors of graduation are either small enough to be neglected, or else, like the errors of observation, devoid of any systematic arrangement. This assumption cannot always be absolutely correct, but no other is eligible when only one series of comparisons has been made.

The following example (Table 1) exhibits a convenient form of record and computation, requiring no tables except Crelle's Rechentafeln, and those contained in these pages. The entire calculation is given here. At

TABLE I.

S.	Verniers. I.	Verniers. II.	R.	D.	$D \sin \frac{1}{2} S$.	$D(1-\cos \frac{1}{2} S)$.	$A \sin \frac{1}{2} S$.	$B(1-\cos \frac{1}{2} S)$.	Ecc. cor.	Ecc. cor. + X.	$O - C$.	$(O-C)^2$.	
°	″	″	° ′ ″	″	″	″	″	″	″	″	″		
0	57	57	0 0 56	+ 26	.00	.00	0.0	0.0	.0	+ 31	− 5	25	Z = 0° 0′ 30″
10	57	63	10 0 60	30	+ 2.61	+ .12	− 3.5	− 0.1	− 3.6	28	+ 2	4	″
20	54	57	20 0 56	26	4.52	.39	7.0	0.5	7.5	24	2	4	− 101.9
30	51	57	30 0 54	24	6.22	.82	10.4	1.1	11.5	20	+ 4	16	− 278.3
40	36	45	40 0 40	10	3.42	.60	13.7	1.9	15.6	16	− 6	36	+ 340.1
50	45	45	50 0 45	15	6.34	1.41	17.0	3.0	20.0	12	+ 3	9	− 40.1 = A
60	36	42	60 0 39	9	4.50	1.21	20.0	4.3	24.3	7	2	4	″
70	36	39	70 0 38	8	4.59	1.45	23.0	5.8	28.8	+ 3	5	25	+ 119.5
80	30	33	80 0 32	+ 2	+ 1.29	+ .47	25.8	7.5	33.3	− 2	4	16	+ 401.4
90	24	27	90 0 26	− 4	− 2.83	− 1.17	28.4	9.4	37.8	6	+ 2	4	− 553.0
100′	3	6	100 0 4	26	19.92	9.28	30.7	11.5	42.2	11	− 15	225	− 32.1 = B
110	9	15	110 0 12	18	14.74	7.67	32.8	13.7	46.5	15	3	9	″
120	0	3	120 0 2	28	24.25	14.00	34.7	16.0	50.7	19	− 9	81	+ 31.2
130	18	24	130 0 21	− 9	− 8.15	− 5.19	36.3	18.5	54.8	− 23	+ 14	196	+ 57.0
140							− 37.7	− 21.1	− 58.8				− 56.7
0	54	27											
				+ 150	+ 33.49	+ 6.47					+ 38	654	+ 31.5 = X
				− 85	− 69.89	− 37.31					− 38	27.05	
												± 5.2	
				+ 65	− 36.40	− 30.84							
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

the time of examination this sextant had just been repaired after its return from sea service.

The first column contains the settings, S, of the sextant, and the fourth gives R, the corresponding readings of the circle, the seconds being the mean of those read from the verniers, and recorded in the two preceding columns. The reading at 0° is the mean of those at the beginning and end of the examination. As the labor of computation is lessened by numerically diminishing the values of D, especially those belonging to the larger angles, Z should be either equal to the circle reading when $S = 0°$, or so chosen as to differ a few seconds therefrom, in the direction, and to an extent, indicated by the subsequent values of R. The differences D in the fifth column are found by (3). Table II is serviceable in filling out

TABLE II.

S.	Sin $\frac{1}{2}$ S.	1 — cos $\frac{1}{2}$ S.
°		
0	.000	.000
10	.087	.004
20	.174	.015
30	.259	.034
40	.342	.060
50	.423	.094
60	.500	.134
70	.574	.181
80	.643	.234
90	.707	.293
100	.766	.357
110	.819	.426
120	.866	.500
130	.906	.577
140	.940	.658
150	.966	.741

the sixth and seventh columns; it should be copied upon a slip of paper in lines spaced like those of the record, and laid upon the latter beside the fifth column, so that each value of D may closely follow its two coefficients. By opening the Rechentafeln at D, the two products can be instantly taken out and entered in the same line. The quantities in each of these three columns are next added, and the respective amounts written underneath.

No material error can be introduced by retaining only two places of decimals in these products. At first view, indeed, even the second place might seem to be superfluous, since D itself is frequently several seconds in error, but in (7) the sums of the two sets of products have coefficients with opposite signs, and the effect of an alteration in one of these sums is, therefore, greater than that of a similar change affecting them both in the same direction. Substituting in (2) the values of A and B from (7):

$$\gamma - \gamma' = \left(-\,1.5671 \sin \tfrac{1}{2}\gamma' + 1.8384\,(1 - \cos \tfrac{1}{2}\gamma')\right) \left[D\right] \tag{10}$$

$$+ \left(7.6467 \sin \tfrac{1}{2}\gamma' - 11.0275\,(1 - \cos \tfrac{1}{2}\gamma')\right) \left[D \sin \tfrac{1}{2} S\right]$$

$$+ \left(-\,11.0275 \sin \tfrac{1}{2}\gamma' + 17.9311\,(1 - \cos \tfrac{1}{2}\gamma')\right) \left[D(1-\cos\tfrac{1}{2}S)\right].$$

The change in the computed correction for eccentricity resulting from any given variation in $[D \sin \frac{1}{2} S]$ will be greatest when the coefficient of the latter in (10) is a maximum—that is, when:

$$7.6467 \cos. \tfrac{1}{2}\gamma' - 11.0275 \sin \tfrac{1}{2}\gamma' = 0,$$

$$\tan \tfrac{1}{2}\gamma' = \frac{7.6467}{11.0275}, \text{ and } \gamma' = 69^\circ\, 29',$$

the coefficient itself being then $+\,2.39$. By similar means it is found that the coefficient of $[D\ (1 - \cos \frac{1}{2} S)]$ attains the numerical maximum $-\,3.12$ when $\gamma' = 63^\circ\, 11'$. Both coefficients reduce to 0 for $\gamma' = 0$. If the twenty-six products are correct to the second decimal place, the limit of this error in the eccentric correction is, therefore, 0 when $\gamma' = 0$, and greatest when γ' is somewhere between 63° and 69°, but everywhere less than $0.''005 \times 13 \times (2.39 + 3.12) = 0.''36$. It will be shown a little farther on that the probable error, due to errors of observation, in the eccentric correction derived from a single series of comparisons, is 0 when $\gamma' = 0$, $0.42\ t$ when $\gamma' = 20^\circ$, and still greater for all larger values of γ', t being the probable error of a single comparison. As t can seldom be much less than 2,″ this probable error is greater than the maximum error in question. It is also apparent that to increase the possible error tenfold, by retaining only one decimal digit in the products, would be unsafe.

The constants A, B, and X are computed in the last column of Table I. Each formula of (7) contains three terms which may be obtained from Table III without any greater inconvenience than that of taking out the tabular products for each digit of the argument separately and adding them together, but the table should be extended, if frequently used. The headings of the columns refer to that argument which is in the same horizontal line, *e. g.*, the third column contains both the term of B having $[D \sin \frac{1}{2} S]$ for its argument, and the term of A for which $[D\ (1 - \cos \frac{1}{2} S)]$ is the argument. The upper sign is to be applied when the argument is positive, the lower if negative. These terms can also be taken from the Rechentafeln, by retaining only the first decimal place of sums containing more than three digits, which involves a maximum error ten-thirteenths of that referred to in connection with the products in columns 6 and 7 of Table I, and otherwise subject to the

TABLE III.

$[D]$				A	B	X	
$[D \sin \frac{1}{2} S]$	A	B		X			
$[D(1 - \cos \frac{1}{2} S)]$		A	B		X		
$\pm$	$\pm$	$\mp$	$\pm$	$\mp$	$\pm$	$\pm$	
$''$	$''$	$''$	$''$	$''$	$''$	$''$	
1	7.647	11.027	17.931	1.567	1.838	.480	1
2	15.293	22.055	35.862	3.134	3.677	.960	2
3	22.940	33.082	53.793	4.701	5.515	1.441	3
4	30.587	44.110	71.724	6.268	7.353	1.921	4
5	38.234	55.137	89.656	7.835	9.192	2.401	5
6	45.880	66.165	107.587	9.402	11.030	2.881	6
7	53.527	77.192	125.518	10.969	12.868	3.361	7
8	61.174	88.220	143.449	12.537	14.707	3.842	8
9	68.821	99.247	161.380	14.104	16.545	4.322	9
10	76.467	110.275	179.311	15.671	18.384	4.802	10

same conditions. Column 8 is expeditiously filled by laying the slip of paper containing Table II upon the form, as before, and opening the Rechentafeln at the value of A; then opening at B, the products in column 9 are obtained with equal facility.

The corrections for eccentricity appear in column 10, each term being the sum of those on the same line in the two preceding columns; these corrections, with the further addition of X, are entered in column 11. After subtracting the quantities in column 11 from the corresponding ones in column 5 the residuals in column 12 remain; their sum should not differ from zero by more than two or three units. Finally the squares of the residuals are entered in column 13, and the probable error of graduation and observation is computed from their sum by (9). The foregoing formulas are equally applicable to any examination consisting of fourteen comparisons at points ten degrees apart upon the arc of the sextant. If it is preferred for any reason to make the comparisons elsewhere than at 0°, 10°, etc., the eccentric corrections obtained for the points compared may be rectified by subtracting from each of them the correction computed for $\gamma' = 0°$.

As a further illustration of what is to expected in practice, the eccentric corrections, constants A and B, residuals, and probable errors, of six sextants, by makers of high repute, are given in Table IV.

TABLE IV.

S.	I.		II.		III.		IV.		V.		VI.	
	Ecc. cor.	*O—C.*	Ecc. cor.	*O—C.*	Ecc. cor.	*O—C.*	Ecc. cor.	*O—C.*	Ecc. cor.	*O—C.*	Ecc. cor.	*O—C.*
°	″	″	″	″	″	″	′ ″	″	′ ″	″	″	″
0	0	0	0	— 2	0	+ 1	0	— 3	0	+ 1	0	— 7
10	0	— 2	+ 1	— 1	+ 3	4	— 4	4	— 8	+ 8	— 7	+ 5
20	0	0	2	0	6	+ 1	9	— 5	16	— 3	13	+ 7
30	0	— 2	3	+ 3	8	— 1	14	+ 9	25	2	17	— 1
40	— 1	+ 3	4	+ 1	9	2	19	11	35	1	19	— 4
50	1	2	6	0	10	9	25	+ 7	45	— 8	20	0
60	1	1	7	+ 4	10	— 3	32	— 7	55	0	18	— 4
70	1	4	8	0	9	+ 3	38	6	1 6	— 4	15	+ 3
80	1	+ 3	9	+ 4	8	+ 8	45	— 3	1 16	— 2	10	+ 2
90	1	— 3	10	— 7	6	— 5	53	+ 6	1 27	+ 3	— 3	— 5
100	1	11	11	— 6	4	+ 11	1 0	+ 1	1 38	6	+ 5	+ 4
110	2	— 2	12	0	+ 1	— 4	1 8	— 11	1 48	+ 18	15	+ 1
120	2	+ 3	14	— 2	— 3	+ 6	1 15	— 6	1 59	— 1	27	— 5
130	2	+ 4	15	+ 6	7	— 8	1 23	+ 12	2 9	— 14	40	+ 2
140	— 2		+ 16		— 12		— 1 30		— 2 19		+ 54	
p. e.		± 3.0		± 2.7		± 4.3		± 5.5		± 5.5		± 3.2
A	— 2.1		+ 11.4		+ 40.4		— 43.3		— 86.1		— 94.6	
B	+ 0.3		+ 7.4		— 76.2		— 75 3		— 88.7		+ 217.3	

The one requiring corrections of more than 2′ is of course exceptional, but no complaint is known to have been made of it before the examination, though it had seen service at sea, possibly in the hands of some scrupulous young officer who carried out his reductions to fractions of a second. It is not the most incorrectly centered instrument in the table, however. The probable error, though derived from too small a number of observations to be regarded as precise, is a useful criterion. When much in excess of 5″ it implies that either the graduation is inaccurate, the telescopic power is too low, the mechanical or optical action of the sextant is imperfect (perhaps, through maladjustment of its parts), the observer is unskillful, or two or more of these unfavorable conditions coexist. Values of less than 3″ are frequently obtained, but this is commonly due in part to an accidental avoidance of the larger errors of observation.

Before proceeding to combine the results of several series of comparisons it is desirable to know the probable error of the corrections deduced from a single series. The eccentric correction for any reading γ'. as expressed in (10), may be written :

$$\gamma - \gamma' = \Sigma \Big[D \Big\{ - 1.5671 \sin \tfrac{1}{2} \gamma' + 1.8384 \, (1 - \cos \tfrac{1}{2} \gamma') + \Big(7.6467 \sin \tfrac{1}{2} \gamma' - 11.0275 \, (1 - \cos \tfrac{1}{2} \gamma')\Big) \sin \tfrac{1}{2} S + \Big(- 11.0275 \sin \tfrac{1}{2} \gamma' + 17.9311 \, (1 - \cos \tfrac{1}{2} \gamma')\Big) \, (1 - \cos \tfrac{1}{2} S) \Big\} \Big].$$

Now if t is the probable error of a single comparison, and consequently of D, the probable error of $\gamma - \gamma'$ will be:

$$t \times \sqrt{\Sigma \left[\left\{ -1.5671 \sin \tfrac{1}{2} \gamma' + 1.8384 (1 - \cos \tfrac{1}{2} \gamma') + \left(7.6467 \sin \tfrac{1}{2} \gamma' - 11.0275 (1 - \cos \tfrac{1}{2} \gamma') \right) \sin \tfrac{1}{2} S + \left(-11.0275 \sin \tfrac{1}{2} \gamma' + 17.9311 (1 - \cos \tfrac{1}{2} \gamma') \right) (1 - \cos \tfrac{1}{2} S) \right\}^2 \right]},$$

the value of which for any given value of γ' may be taken from Table V.

TABLE V.

γ'	Prob. error of ecc. cor.
°	
0	$t \times$ 0.00
10	.23
20	.42
30	.58
40	.71
50	.81
60	.87
70	.90
80	.91
90	.89
100	.86
110	.83
120	.82
130	.85
140	0.94
150	$t \times$ 1.09

The expressions for A, B, and X, (7), are equivalent to:

$$A = \Sigma \left[D \left(-1.5671 + 7.6467 \sin \tfrac{1}{2} S - 11.0275 (1 - \cos \tfrac{1}{2} S) \right) \right],$$

$$B = \Sigma \left[D \left(1.8384 - 11.0275 \sin \tfrac{1}{2} S + 17.9311 (1 - \cos \tfrac{1}{2} S) \right) \right],$$

$$X = \Sigma \left[D \left(0.4802 - 1.5671 \sin \tfrac{1}{2} S + 1.8384 (1 - \cos \tfrac{1}{2} S) \right) \right],$$

from which it is evident that the probable error of A is:

$$t \times \sqrt{\Sigma \left[\left\{ -1.5671 + 7.6467 \sin \tfrac{1}{2} S - 11.0275 (1 - \cos \tfrac{1}{2} S) \right\}^2 \right]} = 1.90\, t,$$

that of B is:

$$t \times \sqrt{\Sigma \left[\left\{ 1.8384 - 11.0275 \sin \tfrac{1}{2} S + 17.9311 (1 - \cos \tfrac{1}{2} S) \right\}^2 \right]} = 4.23\, t,$$

and that of X is:

$$t \times \sqrt{\Sigma \left[\left\{ 0.4802 - 1.5671 \sin \tfrac{1}{2} S + 1.8384 (1 - \cos \tfrac{1}{2} S) \right\}^2 \right]} = 0.69\, t.$$

Different series of comparisons, therefore, cannot be expected to agree very closely in the values of A they furnish, still less in their values of B, but such a relation should nevertheless subsist among these apparently discordant results as to render their respective sets of eccentric corrections reasonably harmonious. The constants A and B are, in fact, as indicated by (1), rectangular coördinates of the index-axis referred to the system in which the center of the sextant arc is the origin, the radius passing through mean $0°$ is the axis of A, and $1''$ of the arc having a radius equal to half the length of the index-bar, from axis to edge of vernier, is the linear unit. By substituting the value of X obtained from the last formula of (6) in the other two, they become the equations in A and B of two right lines, inclined to the axis of A at angles of $121° 35'$, and $124° 44'$, respectively, and determining by their intersection the position of the index-axis. These lines meet at an angle of but $3° 9'$, and since errors of observation can only displace them laterally, without altering their direction,* it follows that, as might be expected, a straight line passing through the index-axis, and perpendicular to the radius through the middle of the arc, is located much more accurately than the position of the axis upon that line.

Let D' be the observed value of D obtained by any one setting S' of the sextant, all the other values of S and D which furnished equations of condition retaining those designations; the local correction of the graduation at S' will then be the difference between D' and the value of D computed by (4) for $S = S'$, or:

$$D' - \left(A \sin \tfrac{1}{2} S' + B\,(1 - \cos \tfrac{1}{2} S') + X\right),$$

which, by substituting the values of A, B, and X, in (7), becomes:

$$\begin{aligned}
&\left(1.5671 \sin \tfrac{1}{2} S' - 1.8384\,(1 - \cos \tfrac{1}{2} S') - 0.4802 + 1\right) D' + \\
&\left(- 7.6467 \sin \tfrac{1}{2} S' + 11.0275\,(1 \cos \tfrac{1}{2} S') + 1.5671\right) D' \sin \tfrac{1}{2} S' + \\
&\left(11.0275 \sin \tfrac{1}{2} S' - 17.9311\,(1 - \cos \tfrac{1}{2} S') - 1.8384\right) D'\,(1 - \cos \tfrac{1}{2} S') + \\
&\left(1.5671 \sin \tfrac{1}{2} S' - 1.8384\,(1 - \cos \tfrac{1}{2} S') - 0.4802\right) \Sigma \left[D \right] + \\
&\left(- 7.6467 \sin \tfrac{1}{2} S' + 11.0275\,(1 - \cos \tfrac{1}{2} S') + 1.5671\right) \Sigma \left[D \sin \tfrac{1}{2} S \right] + \\
&\left(11.0275 \sin \tfrac{1}{2} S' - 17.9311\,(1 - \cos \tfrac{1}{2} S') - 1.8384\right) \Sigma \left[D(1 - \cos \tfrac{1}{2} S) \right].
\end{aligned}$$

*The probable displacement of the line represented by the normal equation in A, which is the one making an angle of $121° 35'$ with the radius through $0°$, is $t \times 0.79$; the probable displacement of the other line is $t \times 0.92$.

The probable error of this correction is therefore:

$$t \times \sqrt{\left\{\begin{array}{l} 1.5671 \sin \tfrac{1}{2} S' - 1.8384\,(1 - \cos \tfrac{1}{2} S') - 0.4802 + 1 + \\ \left(-7.6467 \sin \tfrac{1}{2} S' + 11.0275\,(1 - \cos \tfrac{1}{2} S') + 1.5671\right) \sin \tfrac{1}{2} S' + \\ \left(11.0275 \sin \tfrac{1}{2} S' - 17.9311\,(1 - \cos \tfrac{1}{2} S') - 1.8384\right)(1 - \cos \tfrac{1}{2} S') \end{array}\right\}^2 + \Sigma \left[\left\{\begin{array}{l} 1.5671 \sin \tfrac{1}{2} S' - 1.8384\,(1 - \cos \tfrac{1}{2} S') - 0.4802 + \\ \left(-7.6467 \sin \tfrac{1}{2} S' + 11.0275\,(1 - \cos \tfrac{1}{2} S') + 1.5671\right) \sin \tfrac{1}{2} S + \\ \left(11.0275 \sin \tfrac{1}{2} S' - 17.9311\,(1 - \cos \tfrac{1}{2} S') - 1.8384\right)(1 - \cos \tfrac{1}{2} S) \end{array}\right\}^2\right]},$$

which will be found in the second column of Table VI for all values of S'. When D' belongs not to one of the fourteen comparisons from which A, B, and X were derived, but to an additional comparison at any point whatever, the quantity inclosed by the first parenthesis in the preceding expression retains only the term $+ 1$, and the resulting values are given in the third column of Table VI. In all the foregoing expressions of probable errors $\frac{t}{\sqrt{N}}$ must be substituted for t, if N is the number of series of comparisons constituting the examination.

TABLE VI.

S'	Probable errors of local corrections.	
°		
0	$t \times 0.72$	$t \times 1.22$
10	.85	1.13
20	.91	1.08
30	.93	1.06
40	.93	1.06
50	.92	1.07
60	.92	1.08
70	.92	1.08
80	.92	1.07
90	.93	1.06
100	.93	1.06
110	.91	1.08
120	.85	1.13
130	$t \times 0.72$	1.22
140	----------	1.35
150	----------	$t \times 1.54$

When several series of comparisons are made, they are first reduced separately in the manner already described. It is not absolutely necessary to find the value of X at this stage, or to fill out columns 8 to 13 of the form represented by Table I, but only a little time is saved by these omissions, and useful verifications of the work are lost. In Table VII are given the constants, eccentric corrections, residuals, etc., obtained by the preliminary reduction of three successive examinations of the same sextant.

The means of the three values of A, and B, and the three values of X computed therefrom by (8) are:

$$A = +\ 5''.3 \qquad X' = -\ 24.16$$
$$X'' = -\ 24.94$$
$$B = +\ 68''.0 \qquad X''' = -\ 31.37$$

TABLE VII.

S	FIRST SERIES.		SECOND SERIES.		THIRD SERIES.	
	Ecc. cor.	$O-C$.	Ecc. cor.	$O-C$.	Ecc. cor.	$O-C$.
°	″	″	″	″	″	″
0	0.0	+ 10	0.0	+ 7	0.0	+ 7
10	+ 0.8	− 5	0.0	− 3	+ 1.3	− 4
20	2.0	5	+ 0.6	1	3.1	+ 5
30	3.8	− 7	1.9	2	5.4	− 6
40	5.9	+ 6	3.7	2	8.1	1
50	8.5	− 3	6.2	5	11.2	6
60	11.4	− 10	9.1	− 7	14.7	4
70	14.8	+ 3	12.8	0	18.5	− 1
80	18.4	9	16.8	+ 8	22.7	+ 9
90	22.4	2	21.4	8	27.1	+ 2
100	26.7	1	26.4	3	31.9	0
110	31.2	+ 9	31.9	+ 3	36.8	+ 12
120	35.9	− 6	37.8	− 4	42.0	− 5
130	40.9	− 5	43.9	− 5	47.3	− 6
140	46.0	(− 4)	50.4	(− 15)	52.7	(− 12)
150	+ 51.2	------	+ 57.1	------	+ 58.3	
		″		″		″
Prob. error		± 4.9		± 3.7		± 4.4
	A_1	= + 6.6	A_2	= − 3.5	A_3	= + 12.8
	B_1	= + 60.5	B_2	= + 81.6	B_3	= + 61.8
	X_1	= − 23.2	X_2	= − 23.3	X_3	= − 34.0
	$[D_1]$	= − 103.	$[D_2]$	= − 114.	$[D_3]$	= − 204.
	$[D_1 \sin \frac{1}{2} S]$	= − 1.31	$[D_2 \sin \frac{1}{2} S]$	= − 3.57	$[D_3 \sin \frac{1}{2} S]$	= − 44.12
	$[D_1 (1 - \cos \frac{1}{2} S)]$	= + 13.13	$[D_2 (1 - \cos \frac{1}{2} S)]$	= + 14.04	$[D_3 (1 - \cos \frac{1}{2} S)]$	= − 2.77

The reduction is completed in Table VIII, which is so similar to the corresponding portion of Table I as to require but little explanation.

TABLE VIII.

S	$A \sin \frac{1}{2} S$	$B(1-\cos \frac{1}{2} S)$	Ecc. correction.		D, comp.	D, obs.	$O-C$.	Local cor.	Diff. from mean.	(Diff.)^2
°	″	″	″	″	″	″	″	″	″	
0	0.0	0.0	0.0	0.0	− 24 25 31	− 13 16 27	+ 11 + 9 + 4	+ 8	+ 3 + 1 − 4	9 1 16
10	+ 0.5	+ 0.2	+ 0.7	± 0.2	23 21 31	27 26 37	− 4 − 2 − 6	− 4	0 + 2 − 2	0 4 4
20	0.9	1.0	1.9	0.4	22 23 29	26 24 26	− 4 − 1 + 3	1	− 3 0 + 4	9 0 16
30	1.4	2.3	3.7	0.6	20 21 28	26 23 35	− 6 − 2 − 7	− 5	− 1 + 3 − 2	1 9 4
40	1.8	4.1	5.9	0.7	18 19 25	11 22 27	+ 7 − 3 − 2	+ 1	+ 6 − 4 − 3	36 16 9
50	2.2	6.4	8.6	0.8	16 16 23	18 22 29	− 2 − 6 − 6	− 5	+ 3 − 1 − 1	9 1 1
60	2.6	9.1	11.7	0.9	12 13 20	22 21 23	− 10 − 8 − 3	− 7	− 3 − 1 + 4	9 1 16
70	3.0	12.3	15.3	0.9	9 10 16	6 11 − 16	+ 3 − 1 0	+ 1	+ 2 − 2 − 1	4 4 1
80	3.4	15.9	19.3	0.9	5 6 12	+ 4 + 1 − 2	+ 9 + 7 + 10	9	0 − 2 + 1	0 4 1

90	3.7	19.9	23.6	0.9	1 1 − 8	+ 1 + 6 − 5	+ 2 + 7 + 3	4	− 2 + 3 − 1	4 9 1
100	4.1	24.3	28.4	0.9	+ 4 + 3 − 3	+ 4 + 6 − 2	0 + 3 + 1	1	− 1 + 2 0	1 4 0
110	4.3	29.0	33.3	0.9	+ 9 8 2	+ 17 12 15	+ 8 + 4 + 13	+ 8	0 − 4 + 5	0 16 25
120	4.6	34.0	38.6	0.8	14 14 7	7 10 3	− 7 − 4 − 4	− 5	− 2 + 1 + 1	4 1 1
130	4.8	39.2	44.0	0.9	20 19 13	13 16 7	− 7 − 3 − 6	− 5	− 2 + 2 − 1	4 4 1
140	5.0	44.7	49.7	1.0	26 25 + 18	19 12 + 7	− 7 − 13 − 11	(− 10)	(+ 3) (− 3) (− 1)	
150	+ 5.1	+ 50.4	+ 55.5 ±	1.1						
								+ 32 − 32	+ 43 − 43	260 ″ ± 1.79

The three terms of each group in the fifth column are obtained by successively adding X', X'', and X''', to the eccentric correction in the preceding column. The three values of D following in the sixth column are those which were observed in the first, second, and third, series of comparisons respectively. By subtracting from each of the latter the computed value in the fifth column, the three values of the local correction in the seventh column are found; their mean is given in the eighth column.

The residuals in the ninth column are obtained by subtracting this mean successively from its three constituents, and the tenth column contains the squares of the residuals.

There are 42 residuals, and the sum of their squares is 260; the probable error of a single observation is therefore:

$$t = 0.6745 \sqrt{\frac{260''}{42 - 5}} = \pm 1''.79.$$

The probable errors of the eccentric corrections in the fourth column of Table VIII are taken from Table V with this value of t, and divided by $\sqrt{3}$, since three independent determinations of A and B have been made. The maximum probable error of a local correction deduced from a single series of comparisons, as given in the second column of Table VI, is $t \times 0.93$; the probable error of the local corrections in this example (excepting that derived from the additional comparison at 140°) is therefore not greater than $\frac{\pm 1''.79 \times 0.93}{\sqrt{3}} = \pm 1''.0$, which is small enough to justify some degree of confidence in them. It should be mentioned here that the three series of comparisons were all made with the same portion of the circle in this instance, and that the effect of errors in the circle is consequently but little diminished by the repetition. In a mere illustration of the capabilities of the method this uniformity is preferable, since it affords a value of t nearly identical with that which would be obtained if the circle were faultless, while the absolute verity of the corrections is of minor importance. But charging the sextant with the imperfections of both instruments, and ignoring also the error of observation, which cannot be inappreciable, none of these corrections imply an error of circular division exceeding 5″, one that is certainly to be expected in all graduation except that of the very highest class.

The probable error of observation in this example, $t = \pm 1''.8$, is very small, as it ought to be, for the sextant was firmly supported in a convenient position, the pointing was deliberate, and directed upon a singularly well-defined object, the index was set in a definite position always referred to the same lines of the vernier, and the observer was perhaps somewhat expert at that time. This error will ordinarily be larger, indeed an observer who contents himself with the least count of a 10″ vernier cannot reduce his probable error below 2″.5; but under some circumstances—as, for instance, in measuring circummeridian altitudes of Polaris with the sextant mounted upon a stand—the precision attained in this examination should be closely approached. With practice the vernier may be read as closely as it can be set, for so long as the direction of the necessary movement is recognized, the distance can also be estimated.

The 42 residuals are distributed as follows:

Between	Number of Errors	
	By theory.	Found.
″ ″		
0.0 and 0.5	6	5
0.5 " 1.5	12	12
1.5 " 2.5	9	11
2.5 " 3.5	7	7
3.5 " 4.5	4	5
4.5 " 5.5	2	1
5.5 " 6.5	1	1
6.5 " ∞	1	0

When the preliminary reductions have been carried out in full, as in Table I, the eccentric corrections may also be found by taking the means of the corrections in the different series for the same values of S, and the local corrections by similarly taking the means of the residuals $O - C$; but the final residuals must be obtained as in Table VIII, with values of X', X'', etc., calculated by (8) from the mean values of A and B. Unless this is done the computed probable errors will, in general, be somewhat too small. The effect of local errors upon the determination of eccentricity is usually unimportant. If the eccentric corrections in Table VIII are recomputed after applying the local corrections to the observed values of D, there will be no change amounting to half a second.

Some test of the general trustworthiness of the examination is always desirable. A sextant may be in such condition as to operate correctly under the delicate manipulation it receives upon the table of the apparatus, yet when removed therefrom and handled less cautiously, or returned to its case, a slight displacement of the axis may occur; so that if compared again the two sets of eccentric corrections will differ considerably from each other, although a small probable error is found for each. Any great change of this sort may be detected by comparing the differences D in the successive series of comparisons, which should always be scrutinized for that purpose before beginning the reductions. It is not possible, however, to decide in this way whether small abnormal variations exist or not. The agreement or disagreement between the different pairs of values of A and B is also an insufficient test, for a reason already given, but if the preliminary reductions are carried out far enough to determine the eccentric corrections for each series separately, the probable error of this correction at any point, as deduced from the differences between the corrections furnishedby each of the series and their mean, maybe compared with the same probable error taken from Table V. The two values can scarcely be expected to agree exactly, but the difference between them should not be too great. The following results were obtained from the data in Tables VII and VIII.

γ'.	By diffs.	By Table V.
°	″	″
0	0.0	0.0
20	± 0.9	± 0.8
40	1.5	1.3
70	2.0	1.6
100	2.1	1.5
130	2.2	1.5
150	± 2.6	± 2.0

In reading a sextant it is not merely the coincidence of a single line of the vernier with one of the limb that is noted, but the relative positions of several adjacent lines are taken into account, or ought to be; the effect of errors peculiar to individual lines is thereby rendered comparatively innocuous, for such errors cannot be large without being visible. The most pernicious errors of graduation are progressive displacements in alternating directions, extending throughout the arc in waves more or less regular, but of considerable length. The existence of systematic errors having a period long enough to embrace several of the points which have been examined is indicated by a succession of local corrections with the same algebraic sign. It is sometimes advisable to attempt the correction of such errors, especially when they are large, and when many series of comparisons have been obtained. A convenient process is to plot the values of S as abscissas, and the computed local corrections as ordinates, to draw a fair curve approximating the points thus laid down, and lastly to measure and tabulate the ordinates of the curve as mean local corrections. This method is a rather rough one, but it is useful when the corrections to be adjusted are small, as the local corrections always are. Much exercise of good judgment is, however, essential: if the curve were drawn through all the points, or, what is the same thing, if the computed local corrections were adopted without any adjustment, the error resulting from sporadic defects in the graduation would apparently vanish; but it is not certain that actual errors would always be diminished, for any single local correction may be considerably in error, and may also refer to a point not impartially representing the general state of the graduation in its vicinity. The number of points of contrary flexure in the curve must be very small as compared with the number of given points, and in every doubtful case it is safer to err on the side of proximity to the axis of S. The local corrections in the eighth column of Table VIII are not large, but they show unmistakable signs of systematic arrangement. The mean local corrections in Table IX were accordingly obtained from them by the process just referred to. There is one point of contrary flexure in the curve not far from $S = 68°$.

TABLE IX.

S.	Loc. cor.	Mean loc. cor.	Residuals.
°	″	″	″
0	+ 8	+ 3	+ 5
10	— 4	+ 1	— 5
20	1	— 1	0
30	— 5	3	— 2
40	+ 1	4	+ 5
50	— 5	5	0
60	— 7	— 5	— 2
70	+ 1	+ 1	0
80	9	6	+ 3
90	4	6	— 2
100	1	5	— 4
110	+ 8	+ 2	+ 6
120	— 5	— 1	— 4
130	5	5	0
140	(— 10)	— 10	(0)
			+ 19
			— 19

As this series of supplementary corrections is a somewhat typical one, its significance should be recognized. Disregarding the abberrations of individual lines, the actual and mean arcs are coincident at points near 15°, 68°, and 117°. The mean arc overlaps the actual arc at both ends; the mean length of divisions of the latter is, therefore, too small; they are actually too small between 0° and about 55°, too large from 55° to

85°, and again too small from 85° onward. From a point near 68°, where it is greatest, their length decreases in both directions. These systematic irregularities may have been produced by inequalities in the operation of the dividing engine, but they can also be accounted for by supposing an almost infinitesimal distortion to have occurred after the graduation was executed, the middle of the arc approaching the center, and the ends receding therefrom, the greatest change being at the end opposite 0°. The apparent difference between the two ends may, however, be partially due to a slight deviation from parallelism in the surfaces of the index glass.

TABLE X.

γ	Total correction.
°	″
0	0
10	— 1
20	2
30	2
40	— 1
50	+ 1
60	4
70	13
80	22
90	27
100	30
110	32
120	35
130	36
140	+ 37

The total correction of this sextant, or sum of the eccentric and mean local corrections, is given in Table X. For the sake of convenience in use the correction at 0° has been reduced to 0 by adding — 3″ throughout.

The correction applied to any angle measured with the sextant is always the difference between two tabular corrections—that of the observed reading, and that of the reading made in determining the index correction. Let D_1', D_2', etc., be the observed values of D corresponding to the setting S' in the different series of comparisons, the computed local correction for this reading is then:

$$\left.\begin{array}{l} D_1' - A \sin \frac{1}{2} S' - B\,(1 - \cos \frac{1}{2} S') - X_1 + \\ D_2' - A \sin \frac{1}{2} S' - B\,(1 - \cos \frac{1}{2} S') - X_2 + \\ \quad \text{etc.}, + \text{etc.}, + \\ D_n' - A \sin \frac{1}{2} S' - B\,(1 - \cos \frac{1}{2} S') - X_n \end{array}\right\} \div N =$$

$$\frac{\Sigma\, D' - \Sigma\, X}{N} - A \sin \tfrac{1}{2} S' - B\,(1 - \cos \tfrac{1}{2} S'),$$

and the sum of the eccentric and local corrections is: $\frac{\Sigma\, D' - \Sigma\, X}{N}$. For any other setting S'' this sum is: $\frac{\Sigma\, D'' - \Sigma\, X}{N}$, which subtracted from the preceding expression leaves: $\frac{\Sigma\, D' - \Sigma\, D''}{N}$.

Now the probable error of each of the N differences D_1', D_1'', D_2', D_2'', etc., is t; the probable error of $\frac{\Sigma\, D' - \Sigma\, D''}{N}$ is therefore:

$$\frac{\sqrt{2\, N\, t^2}}{N} = 1.41\, \frac{t}{\sqrt{N}}.$$

But each point of the curve from which the mean local corrections were obtained depends upon two or more of the computed corrections: the probable error of the correction applied to any angular measurement is consequently less than $1.41 \frac{t}{\sqrt{N}}$ in some proportion depending upon the skill with which the curve was traced. This error is still further diminished when the two readings differ so little that their mean local corrections depend in part upon the same comparisons, and it vanishes when that difference is very small. For the series of total corrections in Table X, $1.41 \frac{t}{\sqrt{N}} = 1.41 \frac{\pm 1''.79}{\sqrt{3}} = \pm 1''.45$.

The numerical mean of the residuals in Table IX is $\pm 2''.7$, which is so much greater than $\frac{t}{\sqrt{N}} = \pm 1''.03$, the probable variation due to error of observation, as to excite a suspicion that the sporadic errors of graduation, including systematic errors of short period, are rather large in this instrument.

The vernier is liable to errors as well as the limb, and they are sometimes large enough to require correction. If the initial and terminal lines of the vernier do not simultaneously coincide with lines of the limb the former is usually blamed, though not always justly; for the vernier of Sextant V in Table IV, if of the right length, will afford nearly simultaneous coincidences at the initial line, and at the additional line next the terminal one, on all parts of the arc; while the vernier of Sextant VI in the same table, should apparently be correct near 50°, too long at 0°, and considerably too short at 140°. When the corrections of the limb are known, a better judgment can be formed, but any examination by comparisons of this sort is necessarily limited to the extremities of the vernier, and gives no indication of the state of affairs between them. An examination may be made with the sextant apparatus, however, at any number of points, equidistant or otherwise, by bringing them in succession to the same division of the limb and comparing with the circle.

Let l' be the actual length of a vernier, whose proper or intended length is l,* and whose nominal length, equal to one division of the limb, is i; also let s be the reading at which the vernier is set in making a comparison, r being the corresponding reading of the circle, and z an

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* The lengths l' and l are to be accounted positive when the readings of the limb and vernier increase in the same direction—negative when they increase in opposite directions.

assumed value of that reading when $s = 0$, the true value being $z + x$, in which x is unknown. Then

$$\frac{s}{i}(l - l') = \frac{s}{i}\, l - (z + x - r),^*$$

or if $l - l' = c$, and $\frac{l}{i} = p$,†

$$\frac{s}{i}\, c = ps - z - x + r,$$

and finally by making $r + ps - z = d$,‡ we obtain:

$$\frac{s}{i}\, c + x = d.$$

Each comparison furnishes an equation of this form, and if m comparisons are made, the normal equations are:

$$\left.\begin{aligned} \left[\left(\frac{s}{i}\right)^2\right] c + \left[\frac{s}{i}\right] x - \left[\frac{s}{i} d\right] &= 0, \\ \left[\frac{s}{i}\right] c \quad + \quad m\, x \quad - \left[d\right] &= 0. \end{aligned}\right\} \tag{11}$$

The true reading of a vernier is the product of the actual difference between one of its divisions and a division of the limb, multiplied by the number of divisions embraced in the reading; if then δ' be any reading, δ its corrected or true value, and q the number of divisions in the vernier:

$$\delta = \frac{\delta'}{i}\, q \left(\pm\, i - \frac{l'}{q}\right),$$

the upper and lower signs of i pertaining to the "short" and "long" forms of vernier respectively. The correction is therefore:

$$\delta - \delta' = \frac{\delta'}{i} \Big(i\,(\pm\, q - 1) - l'\Big),$$

or since $i\,(\pm\, q - 1) = l$, and $l - l' = c$,

$$\delta - \delta' = \frac{\delta'}{i}\, c, \tag{12}$$

in which c is positive for either a "short" vernier which is too short, or a "long" vernier which is too long, and vice versa.

Although the normal equations are easily solved for any values of s, time can be saved, even in this case, by adopting a uniform system of

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* In the apparatus here referred to the circle readings diminish as the readings of the usual or "short" form of vernier increase.

† For most sextants $p = 59$.

‡ For a reversed or "long" vernier s is essentially negative.

Table XI.

s	Verniers. I	Verniers. II	r	ps	d	$\frac{s}{i}d$	Correction, $\frac{s}{i}c$		$\frac{s}{i}c+x$	$O-C$	$(O-C)^2$	
′	″	″	° ′ ″	″	″	″	″	″	″	″		
0	51	54	9 59 54	0	− 6	0.0	0.0	0.0	− 3.8	− 2	4	$Z = 10° 0′ 00″$.
1	57	0	9 0 58	0	2	− 0.2	−0.9	± 0.2	4.7	+ 3	9	
2	57	0	8 1 58	0	2	0.4	1.8	0.4	5.6	4	16	+ 42.32
3	54	54	7 2 54	0	6	1.8	2.7	0.6	6.5	+ 1	1	− 51.45
4	48	51	6 3 50	0	10	4.0	3.7	0.8	7.5	− 2	4	− 9.13 = C.
5	48	54	5 4 51	0	9	4.5	4.6	1.0	8.4	1	1	
6	45	48	4 5 46	0	14	8.4	5.5	1 2	9.3	− 5	25	− 29.57
7	51	51	3 6 51	0	9	6.3	6.4	1.4	10.2	+ 1	1	+ 25.75
8	45	48	2 7 46	0	14	11.2	7.3	1.6	11.1	− 3	9	− 3.82 = X.
9	48	48	1 8 48	0	12	10.8	8.2	1 8	12.0	0	0	
10	51	51	0 9 51	0	− 9	− 9.0	−9.1	± 2.0	− 12.9	+ 4	16	
0	54	57										
					− 93	− 56.6				+ 13 − 13	86	

examination and computation. If comparisons are made at the two ends of the vernier, and at nine equidistant points between them, $\frac{s}{i}$ is successively 0, 0.1, 0.2, etc., 1.0; $m = 11$, and the normal equations (11) become:

$$\left.\begin{aligned} 3.85\,c + 5.5\,x - \left[\tfrac{s}{i}d\right] &= 0, \\ 5.5\,c + 11\,x - \left[d\right] &= 0. \end{aligned}\right\} \qquad (13)$$

Their solution gives:

$$\left.\begin{aligned} c &= -\,0.455\left[d\right] + 0.909\left[\tfrac{s}{i}d\right], \\ x &= 0.318\left[d\right] - 0.455\left[\tfrac{s}{i}d\right], \end{aligned}\right\} \qquad (14)$$

and the weight of c is 1.1.

Table XI is an example of the record of examination and form of computation. It refers to a sextant divided to 10′, and reading to 10″ by a "short" vernier, for which, therefore, $i = 10'$ and $p = 59$. The calculation scarcely requires any tables except one of squares for extracting the square root in finding the probable errors, but it is convenient to take from Crelle's table the four products employed in computing the values of c and x by (14).

The first three columns contain the record of examination, and the fourth receives the products ps, when they are expressed. For the sake of clearness the values of r have been given in full, as well as the seconds of ps, but it is unnecessary to write down more than the seconds of r, and the seconds of ps when they vary, as in the case of a sextant divided to 15′ and reading to 15″. A sufficient explanation of the remaining columns is to be found in their headings. The sum of the squares of the residuals being 86, the probable error of a single comparison is $0.6745\sqrt{\frac{86}{11-2}} = \pm\,2''.09$, and that of c is $\frac{\pm\,2''.09}{\sqrt{1.1}} = \pm\,2''.0$. The corrections in the seventh column are computed by making $\delta' = s$, and $i = 10'$, in (12); the appended probable errors are therefore $\pm\,2''.0\,\frac{s}{10'}$.

In the probable error of a single comparison is included the probable error of graduation, and that of observation, and the latter is presumably equal to t if the limb and vernier were examined under similar circumstances. If, therefore, the probable error of one comparison is notably greater than t, there is reason to suspect that the graduation is sensibly imperfect. The probable error of any angular measurement, due to the errors of the vernier corrections, is simply the difference between the probable errors of the two tabular corrections applied, being, in fact, that proportional part of the probable error of c corresponding to the fraction of the vernier actually used. If the residuals indicate the existence of a systematic error, a supplementary correction may be obtained by the graphic process which has been described, though such an adjustment will rarely be required, except when valuable observations have been made with a sextant which had previously received some injury. If the index-bar were bent, for instance, so that the two ends of the vernier are unequally distant from the axis, it will be found that the divisions are longest at the nearer end, and shortest at the more distant one. The vernier corrections, like those pertaining to the limb, may be determined with increased accuracy by several series of comparisons, preferably made with different parts of the circle, and combined by methods too obvious to require special explanation.

For any given sextant reading the argument of the correction is not the reading itself, but that of the point where a line of the vernier coincides with one of the limb*. The readings 8° 59′ 50″ and 9° 0′ 10″ differ only twenty seconds, but upon a sextant divided to 10′ and reading to 10″ they refer to positions on the limb nearly ten degrees apart. This circumstance, which is not invariably mentioned in the text-books, is also of considerable importance in determining eccentric corrections by the methods commonly recommended. When extreme precision is desired, the accuracy of an observation already made may sometimes be increased by a device applicable to any sextant, whether its errors have been investigated or not. Find the point of coincidence of the recorded reading, and, after setting the zero of the vernier exactly upon that position, read the vernier at the other end; then, setting the terminal line of the vernier at the point previously occupied by the initial line, read at the zero end. Subtract the nominal length of the vernier from the sum of the two vernier readings and divide the remainder by three; the quotient is a correction to be applied to the original sextant reading. An error in

* Every sextant reading is the sum of the limb reading and the vernier reading, and may be readily separated into these two parts when the scheme of graduation is known. If the vernier is of the direct or "short" form, almost universally applied to sextants, the point of coincidence may be found by the following rule: To the limb reading add the vernier reading multiplied by the number of divisions in the vernier. Thus on the arc of a sextant divided to 10′, and reading to 10″, and which, therefore, has a vernier of 60 divisions, the reading 6° 49′ 50″ is made at $6° 40' + 9' 50'' \times 60 = 16° 30'$. For a reversed or "long" vernier, the same product is to be subtracted from the limb reading.

the length of the vernier does not affect this result, since one reading is always as much augmented thereby as the other is decreased; but if another limb correction is applied, it should be the mean of the corrections for the three points of coincidence. By this artifice each observation is referred to three distinct positions on the arc, with a corresponding diminution in the effect of purely local errors.

When the presence of large errors upon any small portion of the limb is suspected, in consequence of a local injury or otherwise, as many of the lines in this tract may be examined as the nature of the case requires. These comparisons are not to be employed in finding the eccentricity, but must be reduced separately to determine the local errors.

If it should become generally known that sextants purchased in considerable numbers were inspected only at certain points of the graduation, unscrupulous makers, especially those using copying engines, might be tempted to bestow greater care upon the critical divisions than upon the rest of the arc. But the points to be examined may be selected at pleasure, if their number and distance from each other remain unchanged, as has been shown. In arranging a system of examination for any given service, the number of comparisons in a series will naturally depend on the special requirements of the case. A greater number affords a better representation of the entire graduation, while, on the other hand, it extends the time during which errors may be introduced by gradual or sudden changes in the apparatus, or in the sextant itself. Perhaps there can seldom be any sufficient reason for spacing the comparisons more closely than at points 5° apart.

An alternative form of the apparatus has been devised, having two collimators—one fixed, the other carried by an arm attached to the circle and directed toward the sextant, which is supported by a fixed table immediately over the circle. The two collimator-marks—one seen direct, the other by reflection—are brought into coincidence in the field of the sextant telescope, as in the ordinary use of that instrument. No apparatus of this form has been constructed, but the details have been worked out far enough to show that no serious practical difficulty is to be apprehended. The principal feature of improvement is that nothing will depend upon immobility of the sextant, which lies freely upon its table during the examination and may be removed for inspection at any time. It may be, however, that this advantage is rather apparent than real, for in either case the sextant must be handled with the utmost caution, and experience has not shown that there is any difficulty in preventing an appreciable displacement. On the other hand, the proposed form must necessarily be more expensive than the existing one, the illumination is not so easily effected, and probably the observations will not be quite so good with two collimator images as with one image bisected by a wire. In either form the circle should be provided with microscopes, which are more expeditiously read, and less fatiguing to the eyes, than verniers.

The examples which have been discussed in the preceding pages are sufficient to show that, under favorable circumstances, observations can be made with the sextant leaving very little to be desired as regards accuracy. In practice, however, such precision is not always, perhaps not commonly, attained. The most insidious source of error is an unstable condition of the eccentricity—a fault clearly traceable to defective construction when due attention has been paid to the care and preservation of the sextant. A judicious observer always endeavors to distribute his observations so as to neutralize the unknown errors of his instrument as nearly as possible; but variations in the eccentricity, which may occur at any moment, cannot be evaded by this means. If the conical axis is so improperly fitted as to be circumferentially supported only at the smaller end, its position will probably be maintained by friction, and the viscosity of the wax-like lubricant used for this bearing, until some extraneous force is applied, when displacement into a new position of temporary quiescence may be expected to ensue. Such a movement, to the extent of one-thousandth of an inch in a sextant of seven inches radius, may produce errors of 50″ and upward. Any unavoidable defect in fitting should evidently subsist in the direction opposite to that here supposed, and possibly some changes in the usual dimensions and materials of construction might be advantageous. But whatever the requisite alteration in existing practice may be, its discovery and adoption can safely be intrusted to the instrument-makers if a sufficient inducement to persevere in the search for improvement is offered to them. The mechanician who expends time and money in striving after a perfection which observers do not demand, or appreciate, unwisely impairs his ability to compete with his rivals. When sextants are so generally and adequately tested that the reputation of each maker rests on the actual merits of his work, a remedy for the evils of injudicious design and inferior workmanship will soon be found; until that time it cannot reasonably be expected.