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ERRATUM

Please mark your copy or copies of Smithsonian Contributions to Astrophysics, vol. 5, no. 5 ("On the Motion of Satellites with Critical Inclination"), to effect the following correction:

The illustrations on pages 43 and 49 should be transposed. The illustration on page 49 should go with the legend for figure 1 on page 43, and the illustration on page 43 should go with the legend for figure 3 on page 49.

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ON THE MOTION OF SATELLITES WITH CRITICAL INCLINATION

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Contents

Libration of an Earth Satellite With Critical Inclination

By Yusuke Hagihara [1]

Theories on the motion of an earth satellite have been worked out by Brouwer (1959), Garfinkel (1959), Kozai (1959), Vinti (1959), and others. In all except Vinti's treatment, a divisor of the form $1-5\cos^2 I$ appears, where I denotes the inclination of the satellite's orbit to the earth's equator. Brouwer notes that this method of solution cannot be applied to this case, because the terms containing this divisor become infinitely large for an earth satellite having an orbital inclination with the critical value $\cos^{-1}(1/\sqrt{5})$.

In the present paper I propose to discuss the behavior of the motion of an earth satellite whose orbital inclination is approximately equal to this critical value by applying my general theory of libration (Hagihara, 1944) to the motion of asteroids and natural satellites. We follow Brouwer's notation and begin with his equations of motion, after carrying out a first transformation to eliminate all short period terms in the differential equations. It will be shown by a canonical transformation to these transformed equations that the case for the critical inclination corresponds to the case of double points in a polar diagram in which the radius vector represents $\cos I$ and the argument represents its argument of perigee. This diagram represents the integral of the transformed equations, which are obtained by putting the Hamiltonian function equal to a constant.

When the angular variable describing a motion varies between a limited interval smaller than 2π, and does not reach either the value 0 or 2π, the motion is said to be one of libration. When, on the contrary, it varies from 0 to 2π and makes a complete rotation around the circle, the motion is said to be one of revolution. The motion resembles that of a pendulum hanging from the end of a string whose other end is attached to a fixed support. When the pendulum makes a complete rotation around the supporting point, its motion is one of revolution. When the pendulum returns to the stable equilibrium point in the vertical position below the supporting point, without ever reaching the unstable equilibrium point vertically above its supporting point, and oscillates around the stable equilibrium point, its motion is one of libration.

In the orbit of an earth satellite there are two kinds of double points, a center and a nodal point. Around the double point corresponding to the center, the pericenter of the orbit makes a libration about either of its nodes, ascending or descending. Values of the inclination at either of the nodes, for such a libration, are limited to a narrow interval. An earth satellite having an inclination whose value at the nodes lies outside these narrow limits has a motion of revolution. The boundary of these two kinds of motion consists of those asymptotic motions that approach either of the double points of the second kind corresponding to nodal points, as shown in figure 3.

Equations of the problem

Following Brouwer (1959) we take Delaunay's variables,

$$L=\sqrt{\mu a},$$

$$G=L\sqrt{1-e^2},$$

[1] Smithsonian Astrophysical Observatory.

$H=G\cos I,$

$l=$mean anomaly,

$g=$argument of pericenter,

$h=$longitude of ascending node.

After eliminating all short-period terms by Delaunay's transformation, Brouwer considered the following terms in the Hamiltonian functions:

$$F=F_0+F_1+F_2+\Delta_4F_2,$$

$$F_0=\frac{\mu^2}{2L^2},$$

$$F_1=\frac{\mu^4k_2}{L^3G^3}\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right),$$

$$F_2=\frac{\mu^6k_2^2}{L^{10}}\left[\frac{15}{32}\frac{L^5}{G^5}\left(1-\frac{18}{5}\frac{H^2}{G^2}+\frac{H^4}{G^4}\right)+\frac{3}{8}\frac{L^6}{G^6}\left(1-6\frac{H^2}{G^2}+9\frac{H^4}{G^4}\right)-\frac{15}{32}\frac{L^7}{G^7}\left(1-2\frac{H^2}{G^2}-7\frac{H^4}{G^4}\right)\right]$$
$$+\frac{\mu^6k_2^2}{L^{10}}\left[-\frac{3}{16}\left(\frac{L^5}{G^5}-\frac{L^7}{G^7}\right)\times\left(1-16\frac{H^2}{G^2}+15\frac{H^4}{G^4}\right)\right]\cdot\cos 2g,$$

$$\Delta_4F_2=\frac{\mu^6k_4}{L^{10}}\left[\left(\frac{15}{16}\frac{L^7}{G^7}-\frac{9}{16}\frac{L^5}{G^5}\right)\left(1-10\frac{H^2}{G^2}+\frac{35}{3}\frac{H^4}{G^4}\right)\right]$$
$$-\frac{5}{8}\frac{\mu^6k_4}{L^{10}}\left[\left(\frac{L^7}{G^7}-\frac{L^5}{G^5}\right)\times\left(1-8\frac{H^2}{G^2}+7\frac{H^4}{G^4}\right)\right]\cdot\cos 2g,$$

where we have dropped the primes used in Brouwer's notation for each of the variables; k_2 and k_4 represent the coefficients of expansion of the earth's potential, and μ contains the mass of the earth.

At first we have two integrals,

$$L=\text{constant},$$

$$H=\text{constant},$$

so that we are left with the discussion of the canonical equations

$$\frac{dg}{dt}=-\frac{\partial F}{\partial G},\qquad \frac{dG}{dt}=\frac{\partial F}{\partial g},$$

in which F is of the form:

$$F-F_0=A_1(G)+A_2(G)+B_2(G)\cdot\cos 2g. \tag{1}$$

As F does not contain time t explicitly, we have a further integral,

$$F=\text{constant}, \tag{2}$$

or,

$$A_1(G)+A_2(G)+B_2(G)\cdot\cos 2g=C,$$

and the equations reduce to:

$$\frac{dG}{dt}=-2B_2(G)\cdot\sin 2g, \tag{3}$$

$$\frac{dg}{dt}=-\frac{\partial A_1(G)}{\partial G}-\frac{\partial A_2(G)}{\partial G}-\frac{\partial B_2(G)}{\partial G}\cdot\cos 2g. \tag{4}$$

The motion occurs on the curve representing the integral (2),

$$A_1(G)+A_2(G)+B_2(G)\cdot\cos 2g=C,$$

or, when rewritten,

$$\cos 2g=\frac{C-A_1(G)-A_2(G)}{B_2(G)}, \tag{5}$$

from which we conclude that we must have

$$|C-A_1(G)-A_2(G)|\leqq|B_2(G)|. \tag{6}$$

By combining integrals (3) and (5) we obtain the time t,

$$t-t_0=-\int\frac{dG}{2\sqrt{[B_2(G)]^2-[C-A_1(G)-A_2(G)]^2}}.$$

As the values of G are restricted by the above inequality (6), the ratio $G/H=1/\cos I$ is restricted; hence $\cos I$ is bounded on both sides, and $|\cos I|$ can not reach the value of $+1$, as we shall see later. Thus I oscillates about a certain value I_0. We must find out whether there is a libration for g.

Poincaré transformation

We make the following canonical transformation according to Poincaré:

$$x=\sqrt{G}\cos 2g,$$
$$y=\sqrt{G}\sin 2g.$$

The transformed equations are

$$\frac{dx}{dt}=\frac{\partial F}{\partial y},\ \frac{dy}{dt}=-\frac{\partial F}{\partial x},$$

where

$$F-F_0=\frac{A_1}{2}\left[-1+\frac{3H^2}{r^4}\right]\cdot\frac{H^3}{r^6}$$
$$+A_2\frac{H^5}{r^5}\left[\left\{5-\frac{18H^2}{r^4}+\frac{5H^4}{r^8}\right\}\right.$$
$$-4\left(\frac{L}{H}\right)\frac{H}{r^2}\left\{1-\frac{6H^2}{r^4}+\frac{9H^4}{r^8}\right\}$$
$$\left.-5\left(\frac{L}{H}\right)^2\frac{H^2}{r^4}\left\{1-\frac{2H^2}{r^4}-\frac{7H^4}{r^8}\right\}\right]$$
$$+A_4\frac{H^5}{r^{10}}\left[\left\{\frac{15}{16}\left(\frac{L}{H}\right)^2\frac{H^2}{r^4}-\frac{9}{16}\right\}\right.$$
$$\left.\times\left\{1-\frac{10H^2}{r^4}+\frac{35}{3}\frac{H^4}{r^8}\right\}\right]$$
$$-B_2\frac{H^5}{r^{10}}\left[\left\{1-\left(\frac{L}{H}\right)^2\frac{H^2}{r^4}\right\}\right.$$
$$\left.\times\left\{1-\frac{H^3}{r^4}\right\}\left\{1-\frac{15H^2}{r^4}\right\}\right]\cdot\cos 2g$$
$$-B_4\frac{H^5}{r^{10}}\left[\left\{\left(\frac{L}{H}\right)^2\frac{H^2}{r^4}-1\right\}\right.$$
$$\left.\times\left\{1-\frac{8H^3}{r^4}+\frac{7H^4}{r^8}\right\}\right]\cdot\cos 2g,$$

with

$$A_1=\frac{\mu^4k_2}{L^3H^3},\ A_2=\frac{3}{32}\frac{\mu^6k_2^2}{L^{10}}\left(\frac{L}{H}\right)^5,\ A_4=\frac{\mu^6k_4}{L^{10}}\left(\frac{L}{H}\right)^5,$$
$$B_2=\frac{3}{16}\frac{\mu^6k_2^2}{L^{10}}\left(\frac{L}{H}\right)^5,\ B_4=\frac{5}{8}\frac{\mu^6k_4}{L^{10}}\left(\frac{L}{H}\right)^5,\ r^2=x^2+y^2.$$

If $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=0$, then $x=$constant, $y=$constant, and we have an equilibrium solution which corresponds to the double points of the curve $F(x,y)=C$ representing one of the integrals. Take the first order term F_1 with regard to k_2. We suppose that k_4 is of the same order as k_2^2. Then

$$F_1=\frac{A_1}{2}\left[-1+\frac{3H^2}{(x^2+y^2)^2}\right]\cdot\frac{H^3}{(x^2+y^2)^3}.$$

$F_1=0$ at $x^2+y^2=G=\pm\sqrt{3}\,H$, or $\cos I=\pm 1/\sqrt{5}$, or $I=54^\circ.7$ and $\pi-54^\circ.7$. The variations of G and g are very slow, being of the second order.

Now

$$\frac{\partial F_1}{\partial x}=\frac{3A_1H^3}{(x^2+y^2)^4}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right]\cdot x,$$
$$\frac{\partial F_1}{\partial y}=\frac{3A_1H^3}{(x^2+y^2)^4}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right]\cdot y.$$

$\frac{\partial F_1}{\partial x}$ and $\frac{\partial F_1}{\partial y}$ both vanish at (i) $x=y=0$, (ii) $x^2+y^2=\pm\sqrt{5}\,H$, (iii) $x^2+y^2=\infty$. We shall discuss each of these three cases.

(i) $x=y=0$ has no physical meaning, because $G=0$ and $\cos I=\infty$. $\cos I$ should be bounded by $|\cos I|\leqq 1$, or $G=x^2+y^2\geqq H$. $|\cos I|=1$ corresponds to an equatorial orbit. It does not correspond to a double point.

(ii) $G=\pm\sqrt{5}\,H$ corresponds to Brouwer's critical inclination $\cos I=\pm 1/\sqrt{5}$; that is, $I=63^\circ.4$ and $\pi-63^\circ.4$. Henceforth we suppose that G is always positive and that H is negative when $\cos I<0$ and positive when $\cos I>0$. Then A_1, A_2, A_4, B_2, $B_4>$ or <0 depending on whether $\cos I>$ or <0 and $H>$ or <0. For a retrograde motion of the mean anomaly we suppose $\cos I<0$, so that we always take the mean motion $n_0>0$.

(iii) $x^2+y^2=\infty$ corresponds to a polar orbit with $I=\pi/2$. Since

$$\frac{dF}{dG}=\frac{A_1}{2}\left(-\frac{15H^2}{G^2}+3\right)\frac{H^3}{G^4},$$
$$\frac{d^2F}{dG^2}=3A_1\left(\frac{15H^2}{G^2}-2\right)\frac{H^3}{G^5},$$

we have

$$\frac{d^2F}{dG^2}>0,\ \text{at } G=+\sqrt{5}H,$$

$$\frac{d^2F}{dG^2}<0,\ \text{at } G=-\sqrt{5}H,$$

$$\frac{d^2F}{dG^2}=0,\ \text{at } G=\pm\sqrt{15/2}\ H \text{ and } \infty.$$

Hence F has a minimum at $G=+\sqrt{5}H$ and a maximum at $G=-\sqrt{5}H$, and points of inflection at $G=\pm\sqrt{15/2}H$ and $G=\infty$.

For $G=1/\cos I=x^2+y^2$, $F_1=C_1$ has three roots if

$$0<|C_1|<\frac{|A_1|}{25\sqrt{5}};$$

one root if

$$\frac{|A_1|}{25\sqrt{5}}<|C_1|<|A_1|;$$

and no root if

$$|A_1|<|C_1|.$$

If we take $\cos I=H/(x^2+y^2)$ as the ordinate, the situation is more clearly seen; $\cos I=0$ or $G=\infty$ corresponds to a polar orbit $I=\pm\pi/2$. The values $\cos I=+1$, $G=+H$, or $\cos I=-1$, $G=-H$ both correspond to an equatorial orbit.

Figure 1 shows the curve for $F_1/(A_1/2)$ with G/H as the ordinate; figure 2 shows the same curve with $\cos I=H/G$ as the ordinate. The arrows show the sense of rotation of the pericenter.

Now we have

$$\frac{dg}{dt}=-\frac{\partial F_1}{\partial G}=-\frac{3A_1H^3}{2(x^2+y^2)^4}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right],$$

and dg/dt is positive or negative depending on whether $G^2<5H^2$ or $G^2>5H^2$. Hence the motion of the pericenter is direct for $|\cos I|>1/\sqrt{5}$, that is, for $|\pi/2-I|>26\overset{\circ}{.}6$; and is retrograde for $|\cos I|<1/\sqrt{5}$, that is, for $|\pi/2-I|<26\overset{\circ}{.}6$; and stationary for $|\cos I|=1/\sqrt{5}$, that is, for $|\pi/2-I|=26\overset{\circ}{.}6$. Any point on the critical circle $x^2+y^2=G=\sqrt{5}\,H$ is an equilibrium point; that is, the pericenter is at rest with the fixed value of I but with an arbitrary value of g, the argument of the pericenter.

The circumstance is quite different when we consider the second order terms of F. The orbit is determined by the Hamiltonian

$$F_1=\frac{A_1}{2}\left[-1+\frac{3H^2}{(x^2+y^2)^2}\right]\cdot\frac{H^3}{(x^2+y^2)^3}=C_1,$$

or

$$x^2+y^2=\text{constant}.$$

The motion is given by the equations

$$\frac{dx}{dt}=\frac{3A_1H^3}{(x^2+y^2)^4}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right]\cdot y,$$

$$\frac{dy}{dt}=-\frac{3A_1H^3}{(x^2+y^2)^4}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right]\cdot x.$$

We have

$$y\,\frac{dx}{dt}-x\,\frac{dy}{dt}=\frac{3A_1H^3}{(x^2+y^2)^3}\left[1-\frac{5H^2}{(x^2+y^2)^2}\right].$$

Put

$$x=r\cos\theta,\ y=r\sin\theta;$$

then r^2=constant and

$$\cos^2\theta\cdot\frac{d}{dt}(\tan\theta)=-\frac{3A_1H^3}{r^8}\left[1-\frac{5H^2}{r^4}\right]$$

$$=-2K,\text{ a constant.}$$

Hence

$$\theta=2g=-2K\,(t-t_0),$$

$$g=-K\,(t-t_0).$$

$K=0$ for the critical inclination. We have $G=r^2=H/\cos I$=constant, I=constant, e=constant. Thus, if we take only the first order terms, the orbit is an ellipse with rotating pericenter and rotating node. Further,

$$\frac{dh}{dt}=-\frac{\partial F_1}{\partial H}=-x,\qquad h=-x(t-t_0),\qquad x>0;$$

$$\frac{dl}{dt}=-\frac{\partial F_1}{\partial L}=n_0,\qquad l=n_0(t-t_0),\qquad n_0>0.$$

For the critical inclination the pericenter does not move, whatever g is, but the node rotates with the mean motion,

$$-\frac{3\mu^4k_2}{L^3H^4}(\cos I)^5=-\frac{3}{25\sqrt{5}}\,\frac{\mu^4k_2}{L^3H^4},$$

in the retrograde sense for

$$\cos I=+\frac{1}{\sqrt{5}},$$

and

$$+\frac{3}{25\sqrt{5}}\,\frac{\mu^4k_2}{L^3H^4},$$

in the direct sense for

$$\cos I=-\frac{1}{\sqrt{5}}.$$

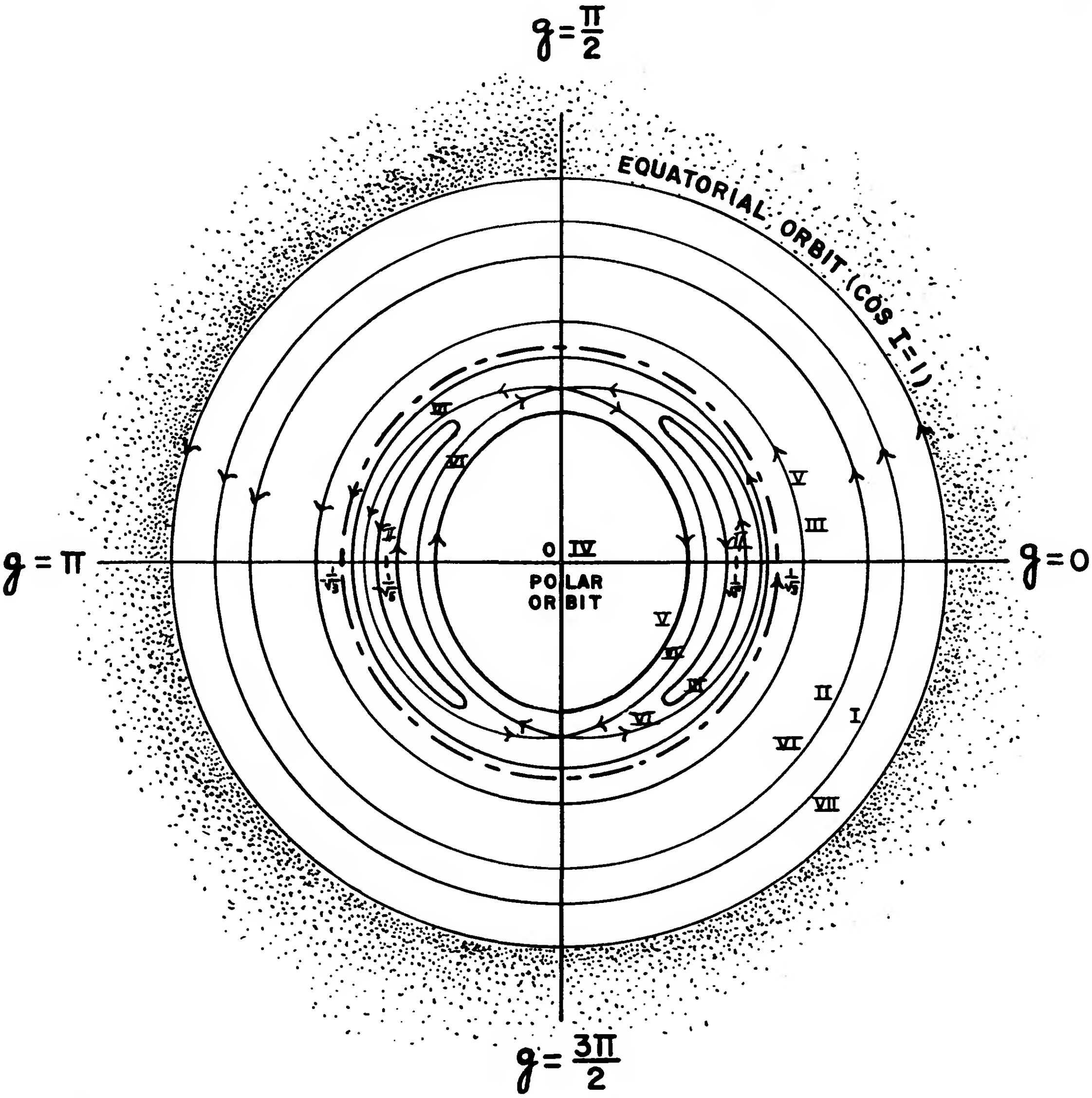

FIGURE 1.—Curves for $F_1/(A_1/2)$ against G/H, where F_1 represents the first order term with coefficient A_1 in the Hamiltonian function and $G/H=1/\cos I$.

First order effects

So far the discussion has dealt with the first order effects in the case in which the earth is supposed to be spheroidal. The center of mass of the earth is taken to be the coordinate origin, so that Brouwer's term Δ_1F_2 does not enter. We next consider the effect of k_2^2 and k_4, that is, the terms F_2 and Δ_4F_2. The position of the pericenter was arbitrary in the first order discussion for the satellite with critical inclination, but when we consider the second order effects the pericenter will be seen to be fixed, as it should be from physical considerations.

We see after some computation that

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=0,$$

at (i)

$$x^2+y^2=\infty\text{, or }\cos I=0,$$

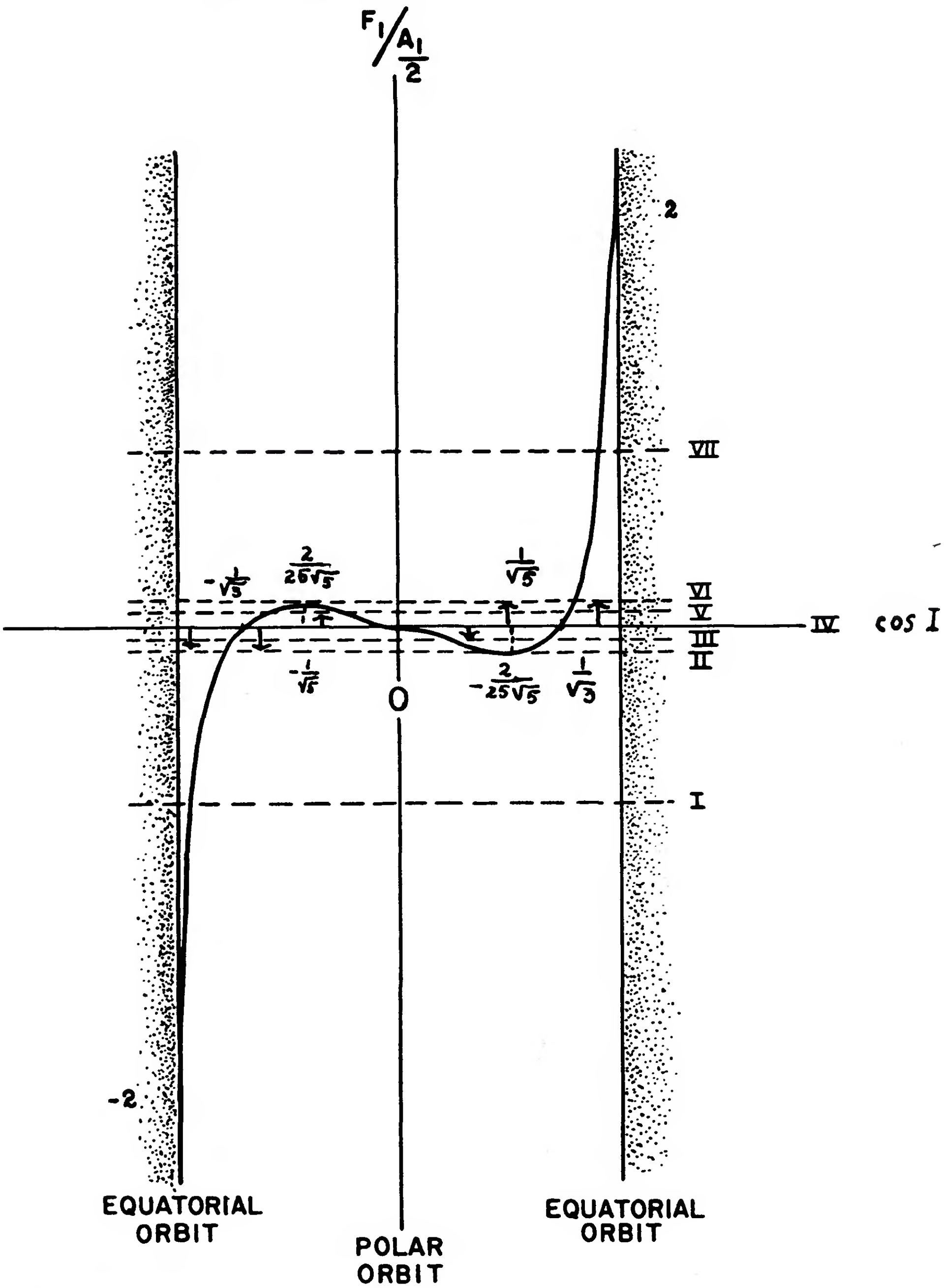

FIGURE 2.—Curves for $F_1/(A_1/2)$ against cos I, where F_1 represents the first order term with coefficient A_1 in the Hamiltonian function.

a polar orbit; and (ii)

$$y=0, \text{ and } \left(\frac{\partial F}{\partial x}\right)_{x/r=\pm 1}=0.$$

The polar orbit

For a polar orbit, for which

$$x^2+y^2=\infty,$$

$$\cos I=0,$$

$$H=G\cos I=0,$$

we have

$$\left(\frac{\partial F}{\partial H}\right)_{H=0}=0;$$

therefore

$$\frac{dh}{dt}=0;$$

$$h=\text{constant}.$$

The node is stationary with an arbitrary position of the node. For $H=0$ we have, by omitting the second order terms,

$$g-g_0=-\frac{3}{2}\frac{\mu^4 k_2}{L^3G^4}(t-t_0).$$

The motion of the pericenter is retrograde for a polar orbit. The eccentricity given by

$$G=L\sqrt{1-e^2}$$

and

$$g+\frac{3}{2}\frac{\mu^4 k_2}{L^3G^4}(t-t_0)$$

oscillates between two finite limits with the period

$$2\pi/\left[\frac{3}{2}\frac{\mu^4 k_2}{L^3G^4}\right].$$

As h is a constant, the node is stationary and the inclination remains constant, and a polar orbit always remains a polar orbit.

Second order effects

In order to examine the case in which

$$y=0$$

and

$$\left(\frac{\partial F}{\partial x}\right)_{x/r=\pm 1}=0$$

we put

$$x^2=\pm\sqrt{5}H(1+\rho)$$

and solve for ρ. Denote the solution for ρ by $\rho=\rho_{\text{I}}$ and $\rho=\rho_{\text{II}}$, respectively, for $x^2=+\sqrt{5}$ H and for $x^2=-\sqrt{5}\,H$. We then obtain the equations

$$30A_1\cdot\rho_{\text{I}}=-A_2\left[\frac{16}{25}\left(\frac{L}{H}\right)^2-\frac{16}{5}\right]$$
$$-A_4\left[33\left(\frac{L}{H}\right)^2+\frac{36}{5}\right]-B_2\left[\frac{152}{25}\left(\frac{L}{H}\right)^2-24\right]$$
$$+B_4\left[\frac{216}{125}\left(\frac{L}{H}\right)^2-\frac{184}{25}\right],$$

$$30A_1\cdot\rho_{\text{II}}=-A_2\left[\frac{16}{25}\left(\frac{L}{H}\right)^2-\frac{16}{5}\right]$$
$$-A_4\left[33\left(\frac{L}{H}\right)^2+\frac{36}{5}\right]+B_2\left[\frac{152}{25}\left(\frac{L}{H}\right)^2-24\right]$$
$$-B_4\left[\frac{216}{125}\left(\frac{L}{H}\right)^2-\frac{184}{25}\right].$$

Thus each of the critical points is displaced by this amount of the first order from the corresponding point in the first order approximation. The critical point should be situated on the x-axis in the second order approximation; that is, the value of θ or $2g$ for the critical point should be 0 or π. Hence the critical points, which were considered to be the equilibrium points in the first order approximation, are unstable except for the point on the x-axis on the critical circle; that is, if we consider the second order terms, they should tend to approach the point on the x-axis; in other words, the pericenter should tend to be on the equator and to coincide with the node.

We consider the equations,

$$C+A_1(r)+A_2(r)+B_2(r)\cos\theta=0,$$

$$\frac{d\theta}{dt}=A_1'(r)+A_2'(r)+B_2'(r)\cos\theta.$$

Here

$$\left|\frac{C+A_1(r)+A_2(r)}{B_2(r)}\right|\leq 1.$$

Hence r, that is, $G=H/\cos I$, is bounded on both sides and I oscillates between finite limits.

$A_1'(r)$ is small near the critical point and is zero at the critical point, and changes sign in passing through it. As $dg/dt=0$ at the critical point, the pericenter is stationary and g is in direct motion for $|\cos I|>1/\sqrt{5}$, that is, for $|\pi/2-I|>26°.6$; and is in retrograde motion for $|\cos I|<1/\sqrt{5}$, that is, for $|\pi/2-I|<26°.6$.

The nearer the pericenter to the critical point, the slower is the motion.

As

$$\frac{dh}{dt}=-3\,\frac{\mu^4 k_2}{L^3H^4}\,(\cos I)^5,$$

the motion of the node is in the retrograde sense near the critical point, $\cos I=+1/\sqrt{5}$, and the direct sense near $\cos I=-1/\sqrt{5}$. The node is stationary for a polar orbit; $|dh/dt|$ increases as I decreases from $\pi/2$ to 0 and is a maximum for an equatorial orbit.

Motion near the critical points

In order to study the nature of the motion near the critical points, we set

$$r^2=x^2+y^2=G=\pm\sqrt{5}H(1+\rho)$$

and suppose that ρ is of the order one-half with regard to k_2. By retaining the terms up to the second order in our integral (2) we obtain

$$C+\frac{A_1}{50\sqrt{5}}[2-15\rho^2]-\frac{A_2}{625\sqrt{5}}\left[40+\frac{16}{\sqrt{5}}\left(\frac{L}{H}\right)-40\left(\frac{L}{H}\right)^2\right]-\frac{8A_4}{1875\sqrt{5}}\left[\frac{15}{16}\left(\frac{L}{H}\right)^2-\frac{9}{16}\right]$$
$$-\frac{8B_2}{125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cos\theta-\frac{8B_4}{1875\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cos\theta=0,$$

$$\frac{d\theta}{dt}=-\frac{6}{25}\frac{A_1}{H}\left[\rho-\frac{11}{2}\rho^2\right]+\frac{A_2}{125H}\times\left[\frac{16}{5}-\frac{16}{25}\left(\frac{L}{H}\right)^2\right]-\frac{A_4}{125H}\left[\frac{9}{20}+\frac{1}{5}\left(\frac{L}{H}\right)^2\right]$$
$$+\frac{B_2}{125H}\left[24-\frac{152}{25}\left(\frac{L}{H}\right)^2\right]\cos\theta+\frac{B_4}{125H}\left[-\frac{184}{25}+\frac{209}{125}\left(\frac{L}{H}\right)^2\right]\cos\theta.$$

First we determine the value of C. For the equilibrium point $P(y=0,\ \cos\theta=+1)$:

$$C_0+\frac{A_1}{25\sqrt{5}}-\frac{A_2}{625\sqrt{5}}\left[40+\frac{16}{\sqrt{5}}\left(\frac{L}{H}\right)-40\left(\frac{L}{H}\right)^2\right]-\frac{8A_4}{1875\sqrt{5}}\left[\frac{15}{16}\left(\frac{L}{H}\right)^2-\frac{9}{16}\right]+\frac{8B_2}{125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]-\frac{8B_4}{3125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]=0.$$

For the equilibrium point $P'(y=0,\ \cos\theta=-1)$:

$$C_0'+\frac{A_1}{25\sqrt{5}}-\frac{A_2}{625\sqrt{5}}\left[40+\frac{16}{\sqrt{5}}\left(\frac{L}{H}\right)-40\left(\frac{L}{H}\right)^2\right]-\frac{8A_4}{3125\sqrt{5}}\left[\frac{15}{16}\left(\frac{L}{H}\right)^2-\frac{9}{16}\right]-\frac{8B_2}{125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]+\frac{8B_4}{3125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]=0.$$

We have

$$\frac{dF}{d\rho}=\frac{3A_1}{5\sqrt{5}}\,\rho=0$$

at $\rho=0$, and

$$\left(\frac{d^2F}{d\rho^2}\right)_{\rho=0}>0,$$

and

$$\left(\frac{d^2F}{d\rho^2}\right)_{\rho=0}<0,$$

respectively, for the points P and P'. Hence F is a minimum at P and a maximum at P'.

We set $C=C_0+\bar{c}_2$, $C=C_0'+\bar{c}_2'$, respectively, for the critical points P and P'. Substituting in equation (2), $F=C$, we get for P:

$$\bar{c}_2-\frac{3A_1}{10\sqrt{5}}\,\rho^2-\frac{8B_2}{125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cdot(1-\cos\theta)+\frac{8B_4}{3125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cdot(1-\cos\theta)=0;$$

for P':

$$\bar{c}_2'+\frac{3A_1}{10\sqrt{5}}\,\rho^2+\frac{8B_2}{125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cdot(1+\cos\theta)-\frac{8B_4}{3125\sqrt{5}}\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]\cdot(1+\cos\theta)=0.$$

To study the behavior of the curve passing through the critical point, we put $\bar{c}_2=0$ and $\bar{c}_2'=0$, respectively, in these two equations. We get no root for ρ in the case $\theta=0+\epsilon$ for P, and two roots for ρ in the case $\theta=\pi+\epsilon$ for P'. Hence P is a center and P' is a saddle point in Poincaré's terminology.

The distance between the two intersections of the x-axis with the two branches passing through P' of the curve represented by the integral (2), that is, $F=C$, gives the width of the domain of libration. By putting $\bar{c}_2'=0$, $\cos\theta=+1$ for this curve, we get:

$$\delta\rho=\frac{4\sqrt{2}}{5\sqrt{5}A_1}\sqrt{-B_2+\frac{B_4}{25}}\cdot\sqrt{\frac{1}{5}\left(\frac{L}{H}\right)^2-1},$$

or

$$\delta I=\frac{4\sqrt{2}}{5\sqrt{(\sqrt{5}-1)A_1}}\cdot\sqrt{-B_2+\frac{B_4}{25}}\cdot\sqrt{\frac{1}{5}\left(\frac{L}{H}\right)^2-1}.$$

Put

$$\frac{6}{25}\frac{A_1}{H}=a_1>0,\quad \frac{3A_1}{10\sqrt{5}}c_1=\bar{c}_1,\quad (i=1,2),$$

and note that

$$\frac{1}{5}\left(\frac{L}{H}\right)^2-1>0$$

and is of the order of the square of eccentricity, as

$$\frac{L}{H}=\frac{G}{\sqrt{1-e^2}\cdot H}=\frac{1}{\sqrt{1-e^2}}[\pm\sqrt{5}(1+\rho)],$$

and provided that

$$\frac{1+\rho}{1-e^2}>1.$$

We assume further that

$$B_2>\frac{1}{5}B_4,$$

and write

$$b_2=\frac{16}{75A_1}\left(B_2-\frac{1}{5}B_4\right)\left[\frac{1}{5}\left(\frac{L}{H}\right)^2-1\right]>0.$$

Then from the equation $F=C$ we obtain for P:

$$\rho^2=c_2-b_2\,(1-\cos\theta);$$

for P':

$$\rho^2=c_2'+b_2\,(1+\cos\theta).$$

A case of libration.—For P we put $b_2-c_2=b_2\cos\theta_0$ and the equation we have to solve is

$$\frac{d\theta}{dt}=-a_1\sqrt{b_2}\sqrt{\cos\theta-\cos\theta_0}.$$

We have

$$-\theta_0\leqq\theta\leqq\theta_0.$$

Hence the pericenter g makes a libration. The amplitude of the libration increases as $(b_2-c_2)/b_2$ decreases. The half-amplitude tends to $\theta_0=\pi/2$, extending to the unstable double points in the limit $(b_2-c_2)/b_2\rightarrow0$ and the motion is asymptotic to these double points. The period of the libration is

$$\frac{T}{2}=\int_{-\theta_0}^{\theta_0}\frac{d\theta}{a_1\sqrt{b_2}\sqrt{\cos\theta-\cos\theta_0}},$$

and becomes infinite in the limit $(b_2-c_2)/b_2\rightarrow0$ (which is a case of asymptotic motion to the critical point), while it tends to zero in the limit $(b_2-c_2)/b_2\rightarrow1$ or $c_2/b_2\rightarrow0$, that is, at the equilibrium point.

We have

$$\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=-\frac{2dz}{\sqrt{1-a}\sqrt{G(z)}}=-a_1\sqrt{b_2}\cdot dt,$$

with

$$\frac{1}{z}=\tan\frac{\theta}{2},\quad a=\cos\theta_0=(b_2-c_2)/b_2,$$

$$G(z)=z^4-\frac{2a}{1-a}z^2-\frac{1+a}{1-a}.$$

According to the formulas in the author's work on the general theory of libration (Hagihara, 1944) we put

$$g_2=\frac{4a^2-3}{3(1-a)^2},\qquad g_3=\frac{(9-8a^2)a}{27(1-a)^3},$$

$$\wp(\epsilon)=\frac{a}{3(1-a)},\qquad \wp'(\epsilon)=0;$$

then we have

$$z=\frac{1}{2}\frac{\wp'u-\wp'\epsilon}{\wp u-\wp\epsilon}=\zeta(u+\epsilon)-\zeta u-\zeta\epsilon,$$

$$du=\frac{dz}{\sqrt{G(z)}}=a_1\frac{\sqrt{c_2}}{2}\cdot dt.$$

As

$$\Delta=g_2^3-27g_3^2=-\frac{1+a}{1-a}\frac{1}{(1-a)^4}<0,$$

$G(z)$ has two real roots, $z_1>0>z_2$, and two conjugate complex roots, and z and $G(z)$ are both real. We take u as the real variable, and

$$2\omega=2\int_{z_1}^{\infty}\frac{dz}{\sqrt{G(z)}}$$

with

$$z_1=\sqrt{\frac{1+a}{1-a}}=\sqrt{\frac{2b_2-c_2}{c_2}},$$

and $2i\omega'$ as the two periods of the elliptic functions. While u varies from 0 to 2ω, t varies from 0 to $\sqrt{c_2/b_2}\cdot\omega$. The period T of the libration with regard to t is $\sqrt{c_2/b_2}\cdot\omega$. Then from the theory of elliptic functions we have:

$$\tan g=\tan\frac{\theta}{2}=2\cdot\frac{\wp u-\wp\epsilon}{\wp' u-\wp'\epsilon}=\frac{1}{\zeta(u+\epsilon)-\zeta u-\zeta\epsilon},$$

$$\zeta u=\frac{\eta u}{\omega}+\frac{\pi}{2\omega}\cot\frac{\pi u}{2\omega}+\frac{2\pi}{\omega}\cdot\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega},$$

with

$$q=\exp\left(-\frac{\pi\omega'}{i\omega}\right),\ \eta=\zeta\omega,\qquad u=\frac{a_1}{2}\sqrt{c_2}\cdot(t-t_0).$$

A case of revolution.—For P' we have $b_2/(c_2'+b_2)<1$ and we have to integrate:

$$\frac{d\theta}{dt}=-a_1\sqrt{c_2'+2b_2}\cdot\sqrt{1-k^2\sin^2\frac{\theta}{2}},$$

$$k^2=\frac{2b_2}{c_2'+2b_2}<1.$$

We have

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\frac{\theta}{2}}}=\frac{dy}{\sqrt{1-y^2}\cdot\sqrt{1-k^2y^2}}=du$$

$$=-a_1\sqrt{c_2'+b_2}\cdot dt,$$

and, by inverting, we get

$$y=\sin\frac{\theta}{2}=\sin g$$

$$=\mathrm{sn}(u,k)=\frac{2\pi}{Kk}\cdot\sum_{n=0}^{\infty}\frac{q^{n+1/2}\sin(2n+1)x}{1-q^{2n+1}},$$

where

$$K=\int_0^1\frac{dy}{\sqrt{1-y^2}\cdot\sqrt{1-k^2y^2}},$$

$$K'=\int_0^1\frac{dy}{\sqrt{1-y^2}\cdot\sqrt{1-k'^2y^2}},$$

$$q=\exp\left(-\frac{\pi K'}{K}\right),$$

$$k^2+k'^2=1,$$

$$u=\frac{2Kx}{\pi}=-a_1\sqrt{c_2'+2b_2}\cdot(t-t_0).$$

Thus the period T with regard to t is

$$T=\frac{4K}{a_1\sqrt{c_2'+2b_2}},$$

or roughly

$$2\pi/[a_1\sqrt{c_2'+2b_2}],$$

as can be seen by expanding the square root in the numerator in powers of k^2, and the mean motion of the pericenter is of the second order.

For $e=0$ we have $L=G=$constant, and $L/H=1/\cos I\approx\pm\sqrt{5}$. In the equation (2), $F=C$, the coefficient b_2 of $\cos 2g$ vanishes and we get $\rho^2=$constant and $d\theta/dt=$constant. The motion of the pericenter is one of revolution. Thus the motion changes from libration to revolution for a nearly circular orbit with vanishingly small eccentricity in the scope of the present approximation.

We have assumed that $B_2>\frac{1}{5}B_4$. If $B_2=\frac{1}{5}B_4$, which, as noted by Kozai (1960), is the case $J_2^2=J_4$ corresponding to Vinti's (1959) assumption; then the term with $\cos 2g$ in the present approximation vanishes, irrespective of the value of the eccentricity. This is a case of revolution, and the question of libration does not arise. If $B_2<\frac{1}{5}B_4$, then libration changes into revolution, and revolution into libration.

Résumé.—To sum up the situation: The pericenter makes a libration about the ascending or the descending node when the orbital inclination is near the critical inclination $I\sim 63.^\circ 4$ or $I\sim 116.^\circ 6$. The position of the pericenter at either of the nodes is stable. The position of the pericenter at the middle point between the two nodes is unstable. The motion of the pericenter is retrograde for

$$\left|\frac{\pi}{2}-I\right|>26.^\circ 6,$$

and is direct for

$$\left|\frac{\pi}{2}-I\right|<26.^\circ 6,$$

and is stationary for

$$\left|\frac{\pi}{2}-I\right|=26.^\circ 6.$$

The situation is illustrated in figure 3, where the Roman numerals along the curves correspond to the value of C illustrated in figure 2.

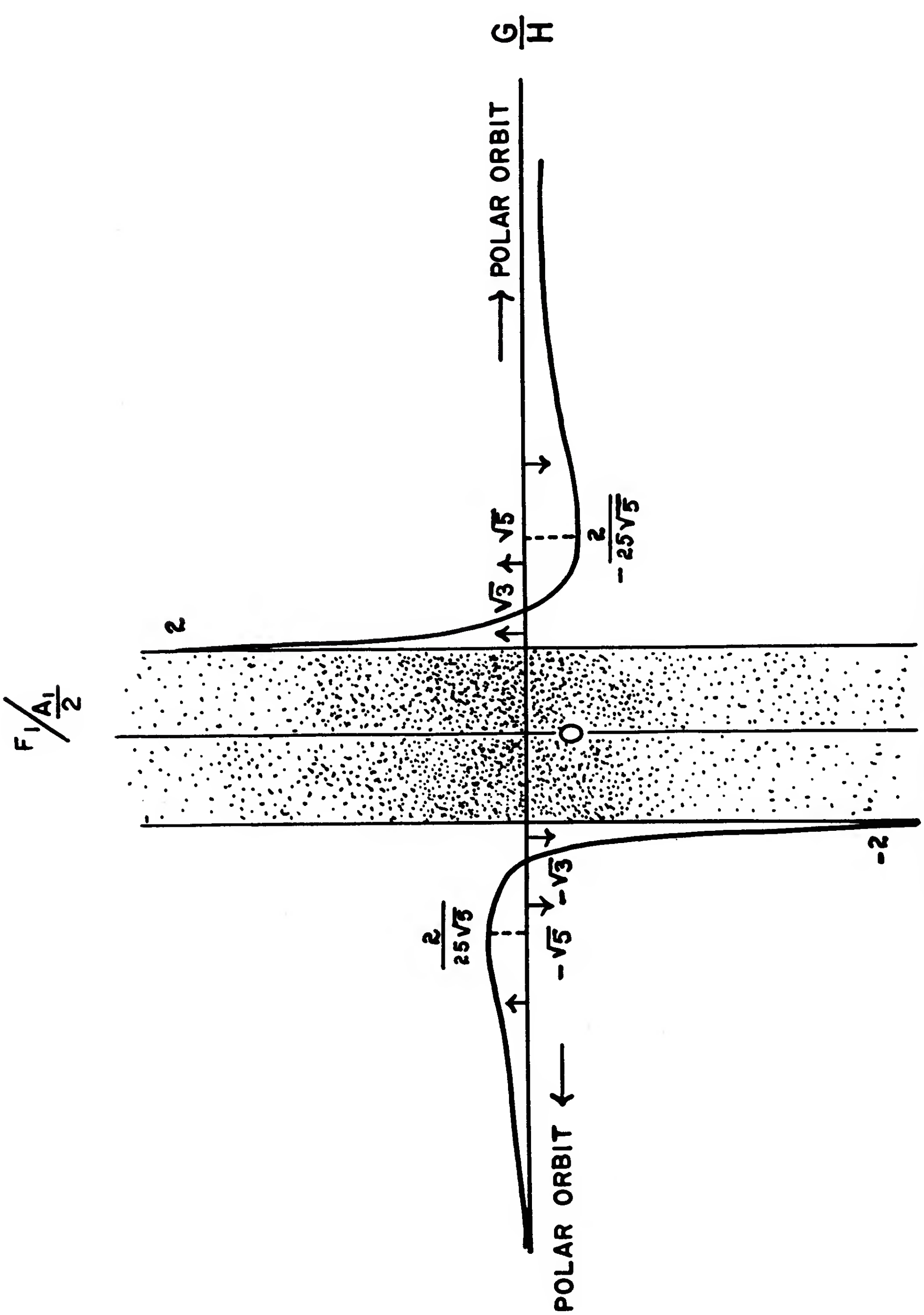

Figure 3.—Curves for the Hamiltonian function $F=C$ drawn with the constant C as the varying parameter.

As G is restricted between two limits, $\sqrt{1-e^2}$ is restricted between two limits. Thus the eccentricity oscillates between two limits and the oscillation has the same period as G or I and g, that is, the period of the libration.

The motion of the node,

$$-3\frac{\mu^4 k_2}{L^3 H^4}(\cos I)^5,$$

is retrograde for $\cos I>0$ and direct for $\cos I<0$. The absolute value of the mean motion of the node is equal to

$$\frac{3}{25\sqrt{5}}\frac{\mu^4 k_2}{L^3 H^4}$$

for the critical inclination. It has a maximum $3\mu^4 k_2/L^3H^4$ for an equatorial orbit.

Effect of the neglected terms

If we take further terms $\Delta_3 F_2$ and $\Delta_5 F_2$ of Brouwer into our discussion, the situation is somewhat different. With the transformation $x=\sqrt{2}G\cos g$, $y=\sqrt{2}G\sin g$, I have worked out the computations to a certain stage. The double points at $g=0$ and $g=\pi$ are modified to be not exactly on the x-axis and we have, as before, the double points $g=\pi/2$ and $g=3\pi/2$, but libration may still occur about the double points near $g=0$ and $g=\pi$ on the critical circle, if the coefficients of the terms containing Brouwer's A_{30} and A_{50}, together with those of the terms considered in the present paper, satisfy certain relations. The term containing A_{30} is the cause of the modification of the double points for $g=0$ and $g=\pi$, because the term vanishes at the critical points. But the question of libration is discussed by the term A_{50}. Let

$$e_2=\frac{\mu A_{50}}{k_2 H^6}\frac{\sqrt{5}}{1875}\sqrt{1-5\left(\frac{H}{L}\right)^2}\times\left[\frac{3}{20}\left\{7-15\left(\frac{H}{L}\right)^2\right\}\right],$$

$$f_2=\frac{\mu A_{50}}{k_2 H^6}\frac{\sqrt{5}}{1875}\sqrt{1-5\left(\frac{H}{L}\right)^2}\times\left[\frac{7}{20}\left\{1-5\left(\frac{H}{L}\right)^2\right\}\right].$$

Then the motion of the pericenter is determined by the equation

$$\frac{dg}{dt}=-a_1\left[(c_2-2b_2)\pm e_2\mp f_2)-(e_2+3f_2)\sin g+2b_2\sin^2 g+4f_2\sin^3 g\right]^{1/2},$$

depending on whether the double point corresponds to $g=\pi/2$ or $g=3\pi/2$. The behavior of the motion near $g=0$ or $g=\pi$ is discussed by the equations

$$\frac{dg}{dt}=-a_1\left[(c_2-b_2)-e_2\sin g+2b_2\sin^2 g-f_2\sin 3g\right]^{1/2},$$

$$\frac{dg}{dt}=-a_1\left[(c_2-3b_2)-e_2\sin g+2b_2\sin^2 g-f_2\sin 3g\right]^{1/2},$$

depending on whether $g\approx 0$ or $g\approx\pi$. A detailed discussion with explicit expressions appears in the succeeding article by Kozai (p. 53).

The author wishes to thank Dr. Kozai for information on the present state of research on the orbits of earth satellites, and for the discussions relating to the present problem.

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Abstract

Brouwer (1959) has noted that for the orbit of an earth satellite with inclination 63°4 with reference to the earth's equator, his solution based on Delaunay's method fails because of the appearance of a factor G^2-5H^2 in the denominator. The present paper shows that this critical inclination corresponds to a case of libration in the ordinary perturbation theory, as an application of the author's general theory of libration in the motion of asteroids and natural satellites. The pericenter of the orbit of an earth satellite having an orbital inclination nearly 63°4 can librate about either of its nodes, ascending or descending, which correspond to the double points of the Hamiltonian function. The motion near the other double points situated at the middle points between the two nodes is one of revolution, and the motion is asymptotic at the double point. For an orbit with this critical inclination, the pericenter is stationary but the node rotates in the retrograde sense when $\cos I>0$ and in the direct sense when $\cos I<0$. However, the libration may change to revolution for an orbit with vanishingly small eccentricity. A polar orbit always remains polar, and the node is stationary but the pericenter moves in a retrograde sense.

Motion of a Particle With Critical Inclination in the Gravitational Field of a Spheroid

By YOSHIHIDE KOZAI[1]

The long-periodic terms in the motion of a particle moving around a spheroid cannot be derived by the usual method of successive approximations if the inclination of the orbit to the equator of the spheroid has the critical value. In this paper the problem is studied by the use of elliptic functions.

The equations of motion

Consider a small particle moving in the gravitational field of a spheroid whose gravitational potential U at a point (r,δ) is expanded into the series of spherical harmonics,

$$U=\frac{\mu}{r}\left\{1+\frac{J_2R^2}{r^2}\left(\frac{1}{2}-\frac{3}{2}\sin^2\delta\right)+\frac{J_3R^3}{r^3}\left(\frac{3}{2}-\frac{5}{2}\sin^2\delta\right)\sin\delta-\frac{J_4R^4}{r^4}\left(\frac{3}{8}-\frac{15}{4}\sin^2\delta+\frac{35}{8}\sin^4\delta\right)\right\},$$

where μ is the gravitational constant multiplied by the mass of the spheroid and R is the equatorial radius of the spheroid. In this potential J_2R^2 is assumed to be a small quantity of the second order, and J_3R^3 and J_4R^4 are of the fourth order.

With Delaunay's canonical variables,

$$L=\sqrt{\mu a},$$

$$G=L\sqrt{1-e^2},$$

$$H=G\cos i,$$

l=mean anomaly,

g=argument of pericenter,

h=longitude of ascending node,

the equations of motion of the particle are expressed in canonical form. In this case the Hamiltonian F depends neither on time explicitly nor on h.

By von Zeipel's (1916) transformation from L, G, H, l, g, and h to L', G', H', l', g', and h', l' can be eliminated from the new Hamiltonian F^* (Brouwer, 1959). Brouwer's expressions of F^* up to terms of the fourth order are the following:

$$\begin{aligned}F^*=&\frac{\mu^2}{2L^2}+\frac{\mu^4J_2R^2}{L^3G^3}\left(-\frac{1}{4}+\frac{3}{4}\frac{H^2}{G^2}\right)\\&+\frac{15}{128}\frac{\mu^6J_2^2R^4}{L^{10}}\left\{\frac{L^5}{G^5}\left(1-\frac{18}{5}\frac{H^2}{G^2}+\frac{H^4}{G^4}\right)\right.\\&+\frac{4}{5}\frac{L^6}{G^6}\left(1-6\frac{H^2}{G^2}+9\frac{H^4}{G^4}\right)\\&\left.-\frac{L^7}{G^7}\left(1-2\frac{H^2}{G^2}-7\frac{H^4}{G^4}\right)\right\}\qquad(1)\\&+\frac{9}{128}\frac{\mu^6J_4R^4}{L^{10}}\left(3\frac{H^5}{G^5}-5\frac{H^7}{G^7}\right)\left(1-10\frac{H^2}{G^2}+\frac{35}{3}\frac{H^4}{G^4}\right)\\&-\frac{3}{64}\frac{\mu^6J_2^2R^4}{L^{10}}\left(\frac{L^5}{G^5}-\frac{L^7}{G^7}\right)\left(1-16\frac{H^2}{G^2}+15\frac{H^4}{G^4}\right)\cos 2g\\&-\frac{15}{64}\frac{\mu^6J_4R^4}{L^{10}}\left(\frac{L^5}{G^5}-\frac{L^7}{G^7}\right)\left(1-8\frac{H^2}{G^2}+7\frac{H^4}{G^4}\right)\cos 2g\\&-\frac{3}{8}\frac{\mu^5J_3R^3}{L^3G^5}\sqrt{1-\frac{G^2}{L^2}}\sqrt{1-\frac{H^2}{G^2}}\left(1-5\frac{H^2}{G^2}\right)\sin g,\end{aligned}$$

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where primes to be attached to the variables are omitted for simplicity.

As F^* depends neither on h nor on l, there exist the following integrals, in addition to an energy integral $F^*=$constant;

$$H=\text{constant},\qquad L=\text{constant}. \tag{2}$$

Then the Hamiltonian F^* is a function of only two variables, G and g, which satisfy the following equations;

$$\frac{dG}{dt}=\frac{\partial F^*}{\partial g}=2Q\sin 2g+S\cos g,$$
$$\frac{dg}{dt}=-\frac{\partial F^*}{\partial G}=-\frac{\partial P}{\partial G}+\frac{\partial Q}{\partial G}\cos 2g-\frac{\partial S}{\partial G}\sin g, \tag{3}$$

where P, Q, and S are functions depending only on G.

By the usual method we integrate these equations approximately by regarding G as a constant and g as a known linear function of time t in the right-hand members.

However, if $H^2/G^2=1/5$, the mean motion of g becomes a small quantity of the fourth order, and the amplitudes of periodic terms in solutions of G and g are of the zero-th order. This means that g and G in the right-hand members cannot be regarded as known functions.

The condition $H^2/G^2=1/5$ is satisfied if the inclination is about 63°.4, which is called a critical inclination.

Equilibrium points

To study the motion of a particle with an inclination very near to the critical one, we may write

$$H^2/G^2=(1+x)/5; \tag{4}$$

thus P, Q, and S are expanded into power series of x, which is regarded as a first order quantity. Expressions of P, Q, and S up to terms of the fifth order are:

$$P(x)=P_0+P_1x+P_2x^2+P_3x^3,$$
$$Q(x)=Q_0+Q_1x,$$
$$S(x)=S_1x,$$

where P_2 and P_3 are of the second order and the other coefficients except for P_0 are of the fourth order. A constant coefficient P_0 will not appear in the following discussion. P_1 is of the second order unless $P(x)$ is expanded around $x=0$. It is remarkable that S_0 vanishes.

Full expressions of P_1, Q_1, and S_1 are as follows:

$$P_1=-\frac{3\mu}{160}\alpha^4\{e^2J_2^2-(28+27e^2)J_4\}\frac{\eta}{a},$$

$$P_2=\frac{3\mu}{16}\alpha^2J_2\frac{\eta}{a},$$

$$P_3=\frac{\mu}{16}\alpha^2J_2\frac{\eta}{a},$$

$$Q_0=\frac{3\mu}{40}e^2\alpha^4(J_2^2+J_4)\frac{\eta}{a},$$

$$Q_1=\frac{3\mu}{40}\alpha^4\left\{J_2^2\left(1+\frac{15}{4}e^2\right)+J_4\left(1+\frac{23}{4}e^2\right)\right\}\frac{\eta}{a},$$

$$S_1=\frac{3\mu}{40}e\alpha^3J_3\frac{\eta}{a},$$

where

$$\mu a=L^2,$$
$$\frac{L^2}{H^2}=\frac{5}{\eta^2}=\frac{5}{1-e^2}=\text{constant},$$
$$\alpha=\frac{R}{a\eta^2}.$$

Here, a and e correspond respectively to the semimajor axis and the eccentricity, but are not equal to the osculating values.

Now the equations to be solved are written as:

$$\frac{dG}{dt}=\{4(Q_0+Q_1x)\sin g+S_1x\}\cos g,$$

$$\frac{dg}{dt}=2\frac{1}{\eta L}(P_1+2P_2x+3P_3x^2-Q_1\cos 2g+S_1\sin g). \tag{5}$$

As the Hamiltonian F^* does not depend on time t explicitly, an equation,

$$F^*=-C, \tag{6}$$

with a constant C, represents a trajectory in the (ξ,η)-plane, where $\xi=(H/G)^2\cos g$, $\eta=(H/G)^2 \sin g$. If derivatives of both G and g with respect to time vanish at a certain point, this is an equilibrium point where the motion in the (ξ,η)-plane is stationary.

There are four equilibrium points of equation (6) in the (ξ,η)-plane, and their coordinates correspond to;

i) $g_1=\kappa, \quad x_1=-\frac{P_1-Q_1}{2P_2}$

ii) $g_2=180°-\kappa, \quad x_2=-\frac{P_1-Q_1}{2P_2}$

iii) $g_3=90°, \quad x_3=-\frac{P_1+Q_1+S_1}{2P_2},$

iv) $g_4=270°, \quad x_4=-\frac{P_1+Q_1-S_1}{2P_2}$

where κ and x_i are of the second order and

$$\sin\kappa=S_1\frac{P_1-Q_1}{8P_2Q_0}.$$

Trajectories in the (ξ, η)-plane

Assume that at a certain epoch $t=t_0$ a trajectory $F^*=-C$ passes through a point corresponding to $x=\delta$ and $g=g_0$; then the energy integral is written as:

$$P(x)-Q(x)\cos 2g+S(x)\sin g$$
$$=P(\delta)-Q(\delta)\cos 2g_0+S(\delta)\sin g_0. \quad (7)$$

Here δ is assumed to be of the second order.

From this integral g is solved as a function of x as follows:

$$\sin g=\frac{-S(x)\pm\sqrt{\{S(x)\}^2-8Q(x)A(x,\delta)}}{4Q(x)}, \quad (8)$$

where

$$A(x,\delta)=P(x)-P(\delta)+Q(x)\cos 2g_0 -S(\delta)\sin g_0-Q(x).$$

Then from the right-hand member of the differential equation,

$$\frac{dx}{dt}=-\frac{1}{L\eta}\{4Q(x)\sin g+S(x)\}\cos g,$$

g can be eliminated so that the equation contains x as the only variable, and the equations become of the following form;

$$\frac{dx}{\sqrt{-f_1(x)f_2(x)}}=\mp\frac{4}{L\eta}dt, \quad (9)$$

where the signs correspond to those of $\cos g$.

If terms of higher order are omitted, $f_1(x)$ and $f_2(x)$ are written as:

$$f_1(x)=P_2x^2-2Q_0\sin^2 g_0, \quad (10)$$
$$f_2(x)=P_2x^2+2Q_0\cos^2 g_0.$$

Here $f_1(x)=0$ corresponds to $g=\kappa$ or $180°-\kappa$, and $f_2(x)=0$ corresponds to $2g=180°$.

If J_2 is positive, P_2 is also positive. Therefore in this case $f_2(x)$ or $f_1(x)$ is positive definite, depending on whether Q_0 is positive or negative.

Then there are two kinds of solutions, both expressed by Jacobi's elliptic functions. If Q_0 is positive, one has

$$x=\sqrt{\frac{2Q_0}{P_2}}\sin g_0\,\mathrm{cn}\{\beta(t-t_0),\sin g_0\},$$
$$\sin g=\sin g_0\,\mathrm{sn}\{\beta(t-t_0),\sin g_0\}, \quad (11)$$

and if Q_0 is negative,

$$x=\sqrt{\frac{-2Q_0}{P_2}}\cos g_0\,\mathrm{cn}\{\beta'(t-t_0),\cos g_0\},$$
$$\cos g=\cos g_0\,\mathrm{sn}\{\beta'(t-t_0),\cos g_0\}, \quad (12)$$

where β and β' are of the third order and $\sqrt{\pm 2Q_0/P_2}$ is of the first order, and

$$\beta=\frac{3}{2}n\alpha^3e\sqrt{\frac{1}{5}J_2(J_2^2+J_4)},$$
$$\beta'=\frac{3}{2}n\alpha^3e\sqrt{-\frac{1}{5}J_2(J_2^2+J_4)}, \quad (13)$$
$$\sqrt{\frac{\pm 2Q_0}{P_2}}=2e\alpha\sqrt{\pm\frac{1}{5}\left(J_2+\frac{J_4}{J_2}\right)}.$$

If Q_0 is positive, maxima and minima of x occur at $g=\kappa$ and $g=180°-\kappa$, and x vanishes at $g=g_0$. On the other hand, if Q_0 is negative, the maxima and minima occur at $g=90°$ and $-90°$.

Trajectories $F^*=-C$ are given in figure 1 for two cases, $Q_0>0$ and $Q_0<0$. If Q_0 is positive and δ is of the second order, g cannot make a complete revolution but has a motion of libration around the equilibrium point near the ξ-axis, unless g_0 is very near to 90° or −90°. However, even though δ is of the first order, g has a motion of libration, if g_0 is very near to κ or $180°-\kappa$. A trajectory with $g_0=\pm 90°$ corresponds to an asymptotic solution.

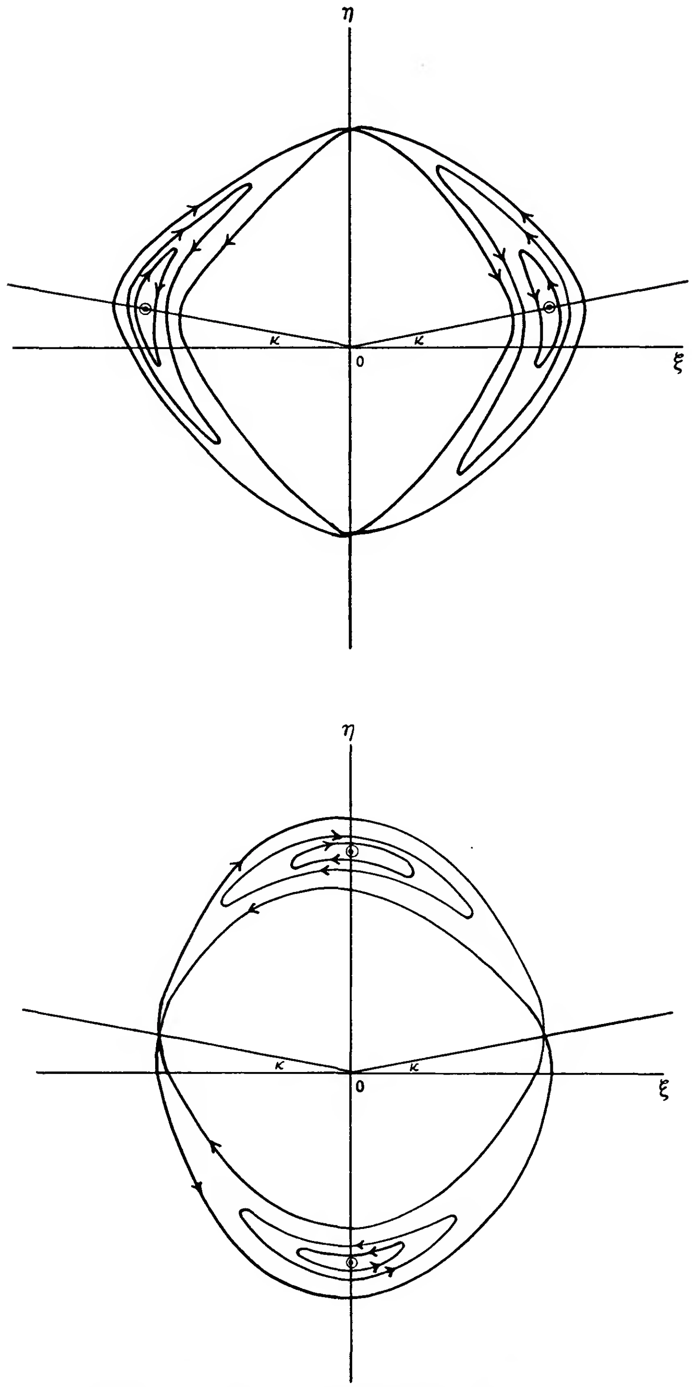

FIGURE 1.—Trajectories in the (ξ, η)-plane. Top: when Q_0 is positive; bottom, when Q_0 is negative.

Trajectories near the equilibrium points

The solutions discussed in the previous section will fail if (i) $|\cos 2g_0 \pm 1|$ is a very small quantity of the second order or if (ii)

$$Q_0 = \frac{3\mu}{40} e^2\alpha^4(J_2^2 + J_4)\frac{\eta}{a}$$

is of the sixth order. In these cases x must be regarded as a quantity of the second order. The condition (ii) means that the trajectory passes very near to one of the four equilibrium points.

Let us assume that $|\cos 2g_0 - 1|$ is very small. In this case, expressions of $f_1(x)$ and $f_2(x)$ are:

$$f_1(x) = 2Q_0$$
$$f_2(x) = P_2X^2 - P_2\epsilon^2 - 2Q_0 \sin^2 g_0, \qquad (14)$$

where

$$x = X - \frac{P_1 - Q_1}{2P_2},$$
$$\delta = \epsilon - \frac{P_1 - Q_1}{2P_2}.$$

If Q_0 is positive, $f_1(x)$ cannot vanish, that is, g cannot reach either to $+90°$ or to $-90°$. And solutions of the equations (9) are,

$$X = \sqrt{\epsilon^2 + 2\frac{Q_0}{P_2}\sin^2 g_0}\ \sin\beta(t-t_0),$$
$$g - g_0 = \sqrt{\frac{P_2\epsilon^2 + 2Q_0 \sin g_0}{2Q_0}}\cos\beta(t-t_0), \qquad (15)$$

where g_0 is κ or $180° - \kappa$ depending on whether $\cos g$ is positive or negative. The angular velocity β has the same expression as that in (13).

If $|\cos 2g_0 + 1|$ is very small, then

$$f_1(x) = 2Q_0,$$
$$f_2(x) = P_2Y^2 - P_2U^2 + 2Q_0 \cos^2 g_0, \qquad (16)$$

where

$$x = Y - \frac{P_1 + Q_1 \pm S_1}{2P_2}, \qquad + \ldots \sin g > 0$$
$$\delta = U - \frac{P_1 + Q_1 \pm S_1}{2P_2}. \qquad - \ldots \sin g < 0$$

Then, if

$$P_2U^2 - 2Q_0 \cos^2 g_0 > 0,$$

$f_2(x)$ has two real roots, and by assuming that Q_0 is positive, one has

$$|Y| = \left(U^2 - \frac{2Q_0 \cos^2 g_0}{P_2}\right)\cosh\beta(t-t_0),$$
$$\cos g = \frac{\sqrt{2(P_2U^2 - 2Q_0\cos^2 g_0)}}{Q_0}\sinh\beta(t-t_0). \qquad (17)$$

In this case any trajectory will cut the ξ-axis, and tends to make a complete revolution around the origin. On the other hand, if

$$U^2 - \frac{2Q_0}{P_2}\cos^2 g_0 < 0,$$

the trajectory cannot cut the ξ-axis, and

$$Y = \left(\frac{2Q_0\cos^2 g_0}{P_2} - U^2\right)\sinh\beta(t-t_0),$$
$$\cos g = \frac{\sqrt{2(Q_0\cos^2 g_0 - P_2U^2)}}{Q_0}\cosh\beta(t-t_0). \qquad (18)$$

The solutions (17) and (18) containing hypertrigonometric terms are valid only if the distance to one of the equilibrium points is of the second order.

Solutions when Q_0 is negative should be derived in a similar way.

Small eccentricity

If Q_0 is very small the problem becomes very complicated, because in this case J_2^3, J_2J_3, and J_2J_4 terms must be added to the Hamiltonian. However, if Q_0 is very small because of a factor e^2, similar coefficients in the J_2^3, J_2J_3, and J_2J_4 terms are also very small, due to the same factor e^2, and are neglected.

In this case κ is of the first order, and

$$-f_1(x) = P_2x^2 + (P_1 - Q_1)x - P_2\delta^2 - (P_1 - Q_1)\delta - 2(Q_0 + Q_1\delta)\sin^2 g_0,$$
$$f_2(x) = P_2x^2 + (P_1 + Q_1)x - P_2\delta^2 - (P_1 + Q_1)\delta + 2(Q_0 + Q_1\delta)\cos^2 g_0. \qquad (19)$$

The discriminants D_1 and D_2 of the quadratic equations $f_1(x) = 0$ and $f_2(x) = 0$ are;

$$D_1 = (P_1 - Q_1 + 2P_2\delta)^2 + 8P_2(Q_0 + Q_1\delta)\sin^2 g_0,$$
$$D_2 = (P_1 + Q_1 + 2P_2\delta)^2 - 8P_2(Q_0 + Q_1\delta)\cos^2 g_0. \qquad (20)$$

Therefore, at least one of D_1 and D_2 is positive.

If D_1 is positive and D_2 is negative, g has a motion of libration around the equilibrium point near the ξ-axis, and the amplitude of x is of the second order. If D_1 is negative and D_2 is positive, the libration will occur around the equilibrium point on the η-axis.

However, if both D_1 and D_2 are positive, g has a motion of libration around one of the equilibrium points, if the two roots of one equation $f_1(x)=0$ are both larger than or both less than the two roots of $f_2(x)=0$. Otherwise, g can make one complete revolution, and x is periodic with an amplitude of the second order.

Node and mean anomaly

The longitude of the node is derived by the equation,

$$\frac{dh}{dt}=-\frac{3}{2}\frac{1}{\sqrt{5}}\left(1+\frac{5}{2}x\right)J_2\alpha^2 n. \qquad (21)$$

If Q_0 is positive and x is expressed by equation (11), then

$$\sin\left\{-\frac{\sin g_0}{\sqrt{5}}(h-h_s)\right\}=\sin g_0\,\mathrm{sn}\{\beta(t-t_0),\sin g_0\}, \qquad (22)$$

where h_s is the secular part of h.

If Q_0 is negative, $\sin g_0$ in equation (22) must be replaced by $\cos g_0$.

Similar results are obtained for the mean anomaly. If Q_0 is positive, the solution is:

$$l-l_s=\frac{1}{\sqrt{5}}\alpha\frac{1}{e}\eta^3\sqrt{J_2+\frac{J_4}{J_2}}\,\mathrm{E}\{\beta(t-t_0),\sin g_0\}, \qquad (23)$$

where l_s is the secular part of l and E is an elliptic integral of the second kind.

The solutions in the present paper are similar to those for the commensurable case for the characteristic asteroids (Hagihara, 1944), but are more simple than the asteroidal case.

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NOTE ADDED IN PROOF: After this manuscript had been sent to the printer, G. Hori's paper entitled "The Motion of an Artificial Satellite in the Vicinity of the Critical Inclination" was published in the Astronomical Journal (vol. 65, pp. 291–300, 1960). His theory also is based on Brouwer's (1959) expression of F^*; however, he follows rather different procedures.

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Abstract

The motion of a particle with a critical inclination is treated on the basis of Brouwer's expression of the transformed Hamiltonian F^*. In the potential of the central spheroid, it is assumed that the second harmonic is of the second order, and that the third and the fourth harmonics are of the fourth order. The solutions are classified into three types: the case of libration around one of four equilibrium points, the case of a complete revolution, and the asymptotic case.

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