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DYNAMICAL STABILITY OF AEROPLANES

(With Three Plates)

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PART I. LONGITUDINAL MOTION

§1. INTRODUCTION AND CONCLUSIONS

The present dynamical investigation of the stability of motion of aeroplanes is based upon the well-known theory of small oscillations of rigid dynamics as first applied by Bryan[1] to aeroplanes and extended by Bairstow.[2] The necessary coefficients for use in the equations of motion were determined by model tests in the wind tunnel of the Massachusetts Institute of Technology.

The application of model experiments to predict the performance of full-size aeroplanes is now so well established that no general discussion of the theory of models is undertaken. A great part of the actual experimental work was performed by Messrs. Huff and Douglas. The oscillating apparatus was designed by Mr. Chow under the direction of Professor E. B. Wilson of the Department of Mathematics. Captain V. E. Clark, U. S. A., while a student in aeronautical engineering, designed an aeroplane which was selected as one type for investigation.

It is necessary to acknowledge the cordial interest taken in the work by Professor C. H. Peabody, head of the Department of Naval Architecture. From the beginning of aeronautical research in his department, Professor Peabody has offered the warmest encouragement by countless arrangements to facilitate progress and to prevent interruptions.

Following the analysis of Clark's aeroplane, the work was repeated for a model of a military aeroplane known as Curtiss JN2.[3] The

[1] G. H. Bryan, "Stability in Aviation."

[2] Technical Report of the Advisory Committee for Aeronautics, London, 1912-13.

[3] Plans and description given in "First Annual Report of the National Advisory Committee for Aeronautics" (Report No. 1, "Report on Behavior of Aeroplanes in Gusts," by J. C. Hunsaker and E. B. Wilson, Washington, D. C., 1916), Senate Document No. 268, 64th Cong., 1st Sess.

Curtiss Aeroplane Company gave their full cooperation with a desire to learn what improvements in the design might be suggested by our stability calculations. Dr. A. F. Zahm of the research department of that company made careful tests to locate the center of gravity and to determine the moments of inertia of the actual aeroplane.

The Curtiss machine is a practical aeroplane with powerful controls, which does not pretend to possess any particular degree of stability. The Clark aeroplane, on the other hand, was designed to be inherently stable while departing as little as possible from the lines of the ordinary military aeroplane as typified by the Curtiss JN2.

The comparison of these two aeroplanes is interesting and leads to the conclusion that inherent dynamical stability, both longitudinal and lateral, may be secured in an aeroplane of current type by careful adjustment of its surfaces and without material effect on controllability or performance.

The discussion in detail is confined to the Clark model, for brevity of presentation, and the results only of the parallel calculations for the Curtiss model are introduced where a comparison is suggested.

In Part I the general equations of motion are deduced in a simplified form which applies to horizontal flight in still air. The longitudinal motion is then considered separately and the necessary aerodynamical constants determined from wind tunnel tests. It is found that the longitudinal motion, if disturbed by any accidental cause, is a slow undulation involving a rising and sinking of the aeroplane as well as a pitching motion. This undulation is stable for high aeroplane speeds since it is rapidly damped out. At lower speeds, the undulation is less heavily damped until at a certain critical low speed the damping vanishes. For speeds below this critical speed, the undulations tend to increase in amplitude with each swing and the longitudinal motion is, therefore, unstable.

Similar calculations for the Curtiss aeroplane show a similar critical speed below which the longitudinal motion is unstable. It is believed that the existence of instability at low speeds has not been indicated before, and it is hoped that the recommendations made to reduce the critical speed may be of assistance to designers.

It appears a simple matter to secure any desired degree of longitudinal stability by the use of properly inclined tail surface, and by the use of light wing loading. It is pointed out that excessive *statical* stability, as indicated by strong restoring moments, is undesirable and may cause the motion to become violent in gusty air. This violence of motion may seriously impair the pilot's control and the aeroplane may "take charge" at a critical time.

However, the longitudinal motion for any particular speed of flight may be made dynamically stable, while at the same time only slightly stable in the static sense, by the use of a large tail surface which lies very nearly in the relative wind. If the minimum of statical stability be combined with the maximum of damping, the pitching will be very slow and heavily damped. The longitudinal motion can then be dynamically stable and yet be without violence of motion in gusty air.

The general prejudice among pilots against "very stable" aeroplanes is believed to be justified. It cannot be too strongly insisted upon that true dynamical stability is better given by damping than by stiffness.

Experience with rolling of vessels has led to the design of vessels of small metacentric height (a measure of statical stability) fitted with generous bilge-keels (damping surface) for passenger carrying. An effort is made to get away from the violence of motion associated with stiffness.

In Part II, the lateral or asymmetrical motion is investigated. The necessary aerodynamical constants are determined by wind tunnel tests wherever practicable and two coefficients which cannot readily be found experimentally are calculated by a simple approximate method. The character of the motion as indicated by the solution of the determinant formed from the equations is then discussed.

It is found that the lateral motion is a combination of a roll, yaw, and side slip or "skidding." Using approximate methods, the combined motion is resolved into three components, two of which are important.

One type of motion is a spiral subsidence if stable or divergence if unstable. The Clark aeroplane becomes spirally unstable at low speed. The motion is a "spiral dive" due to an overbank and a side slip inwards. The aeroplane makes a rapid turn with rapidly increasing bank accompanied by side slipping inwards. The instability is such that an initial deviation from course will double itself in about 7 seconds.

It is shown that the spiral motion may be made stable by adequate fin surface above the center of gravity or upturned wings and by reduction in "weather helm" due to too much rudder or fin surface aft.

The Curtiss aeroplane shows the same sort of spiral instability at high speeds. This aeroplane had no dihedral angle of wings and had a large rudder and deep body.

The second type of motion has been called a "Dutch roll" from analogy to a figure in ice skating. The aeroplane takes up an oscillation in yaw and roll simultaneously. It swings to the right banking for a right turn, then swings back to the left banking for a left turn. The combined yaw and roll has a fairly rapid period. The Clark model at all speeds shows that this motion is heavily damped and hence stable. At high speed, the period is 6 seconds and an initial amplitude is damped to half value in less than 2 seconds. At low speed the period is 12 seconds, damped to half amplitude in 6 seconds.

It appears from an approximate calculation that the "Dutch roll" may become unstable if an aeroplane has too much high fin surface and if there be not sufficient "weather helm" or rear fin surface. It is seen that these conditions are the reverse of those for spiral instability. The conflicting nature of the requirements for stability in these two kinds of motion suggests that an aeroplane is unlikely ever to be unstable in each sense. It also indicates the difficulty of obtaining lateral stability by raised wing tips.

In confirmation of theory, we found the Curtiss spirally unstable at high speed and stable in the "Dutch roll," while at low speed the spiral motion was stable and the "Dutch roll" unstable. The period was 6 seconds and an initial amplitude doubled itself in 8 seconds.

It is believed that the majority of modern aeroplanes of ordinary type are spirally unstable because of excess of fin surface aft. When attempts have been made to remedy this fault by use of a large dihedral angle upwards for the wings, matters have been made worse. It is only to be expected that in overcorrecting spiral instability a "Dutch roll" of more or less violence may be introduced. Especially in gusty air would one expect high fin surface to produce violent rolling.

The Clark aeroplane has very slight rise of wings, about 1°6, and a small rudder. It is shown that at ordinary speeds this aeroplane is stable in every sense, both longitudinally and laterally. Whether this stability is excessive can only be determined by actual flight experience in turbulent air. However, it has appeared possible to secure a degree of stability in every sense in an aeroplane of conventional type.

The object of the research has been to show how various features of design may affect the motion of the aeroplane and only incidentally to produce a stable aeroplane. The discussion has been confined to motion in still air. If an aeroplane be unstable in still air it is obviously worse off in gusts. The converse is, unfortunately, not true, for an aeroplane which is very stable in still air may be so stiff that in turbulent air it will be violently tossed about.

It is conservative to conclude that aeroplanes should not be unstable and that they need not be, since slight changes in the nature of adjustments suffice to correct such instability of motion.

In view of the military use of aeroplanes inside the zone of fire the probability of controls becoming inoperative is ever present. An inherently stable aeroplane, with controls abandoned or shot away, could still be operated by a skilful pilot by manipulation of the motor power alone.

Any sort of automatic (or gyroscopic) stabilizer which operates on the controls is of no use when those controls fail, and it should be judged as an accessory to assist a pilot rather than as a cure-all for the inherent instability of an aeroplane's motion.

The ordinary type of aeroplane readily lends itself to adjustments which make for inherent stability of motion and there is no reason to seek radical changes of type to insure stability. Freak aeroplanes of great "stability" may be excessively stable in some ways and frankly unstable in others. It is likely that the most satisfactory aeroplane will be only slightly stable and that this aeroplane will in any possible attitude be easily controlled by the pilot.

Controllability and statical stability are to some extent incompatible. Dynamical stability requires some amount of statical stability and considerable damping. It appears to be of advantage to provide the minimum of statical stability and the maximum of damping. Then the aeroplane's motion will be of very long period but heavily damped.

It is believed that the methods of investigation here described may be applied to any type of aeroplane, and, by the systematic variation of one feature of design at a time, a full understanding may be had of the effect on the motion of each change. The process is of necessity laborious, but compared with the difficulty of full-scale experiment in the open air, the model method is rapid and inexpensive. It is rarely possible in actual flying to obtain any idea of the effect of slight changes in design. Weather conditions, motor troubles, personal peculiarities of pilots, etc., tend to add to the complexity of an otherwise very simple problem.

Furthermore, experimental flying is dangerous. For example, I knew a pilot who, to determine whether a new aeroplane was spirally unstable, took his machine up to a good altitude and allowed it to get into a spiral dive. The machine made five turns of a rapidly winding and contracting helix before he could bring it out on a horizontal path. If the controls had been only a little less powerful the machine would surely have crashed to the ground. That the controls were adequate was purely a matter of good fortune. The experiment was a success in that spiral instability was demonstrated. Only a few minutes of time was required. However, no information was obtained as to the degree of instability present nor as to what particular changes would remedy matters. To complete the experiment, it would be necessary to repeat the dangerous feat for every change which suggested itself. Naturally, a designer will be very economical in his suggestions under such conditions.

§2. TYPE DESIGN

The type aeroplane selected for the first investigation is a two-place biplane tractor designed by Captain V. E. Clark, U. S. A., while a student in the graduate course in aeronautical engineering at the Massachusetts Institute of Technology. This aeroplane is considered to be representative of modern practice in aeroplane design. Its principal dimensions are as follows:

Wing area, including ailerons	464	sq. ft.
Span, feet	41	max., 40.2 mean.
Aspect ratio	7	
Gap	6.37	ft.
Dihedral of wings, degrees	176.75	
Area, stabilizer	16.1	sq. ft.
Area, elevators	16.0	sq. ft.
Area, rudder	9.35	sq. ft.
Length, body	24.5	ft.
Depth, body, maximum	3.2	ft.
Width, body, maximum	3.3	ft.
Weight, bare	1,070	lbs.
Weight, personnel	320	lbs.
Weight, fuel and oil	415	lbs.
Weight, full load	1,805	lbs.
Radii of gyration	5.2	ft., in roll.
	4.65	ft., in pitch.
	6.975	ft., in yaw.
Brake horse-power	110	
Fuel and oil per B. H. P., hour	0.73	lb.

Maximum speed	87	miles per hour.
Minimum speed	35	miles per hour.
Initial rate of climb	900	ft. per min.
Best glide	1 in 9	
Endurance, full power	5.6	hours.
Endurance, reduced power, 14 hours at	47	miles per hour.

§3. MODEL

A model, $\frac{1}{26}$ scale, was made by Edward Tighe, model maker, giving a span of 1.58 feet. The size of the model was limited by the size of the wind tunnel which is 16 square feet in section. The model is shown in figure 1 (see pp. 8 and 9). Note that wires are omitted and struts are made round instead of "stream line" in section. It is believed that the effects of these changes on total resistance largely counterbalance each other. This model was carefully shellacked and polished to minimize skin friction. The model is, of course, much more smooth than the full-size aeroplane, as it should be, in order that the surfaces may remain geometrically similar. Model work was to the nearest hundredth of an inch. No propeller was fitted, but in the design account was taken of the propeller race in augmenting resistance.

For simplicity, the model was made with trailing ailerons or wing flaps integral with the wings. This somewhat increases the effective supporting area. The stabilizer and elevator were made in one, corresponding to the elevator flaps in neutral position. These points are made clear on figure 1.

§4. WING COEFFICIENTS

In the course of the design, a wing section was devised by Clark which showed a low resistance at high speed and small angle of attack and at the same time was thick enough to permit the use of robust wing spars. A model of this wing was made, of 18 inches span by 3 inches chord, and tested in the wind tunnel. For various angles of wing chord to wind, the lift L, drift D in pounds, and pitching moment M in pounds-inches were observed for a wind of 30 miles per hour; air of density .07608 pound per cubic foot.

The wind tunnel and balance are duplicates of the 4-foot installation at the National Physical Laboratory, England, and reference may be made to the technical report of the Advisory Committee for Aeronautics, year 1912-13, for a description of the apparatus and method of operation.

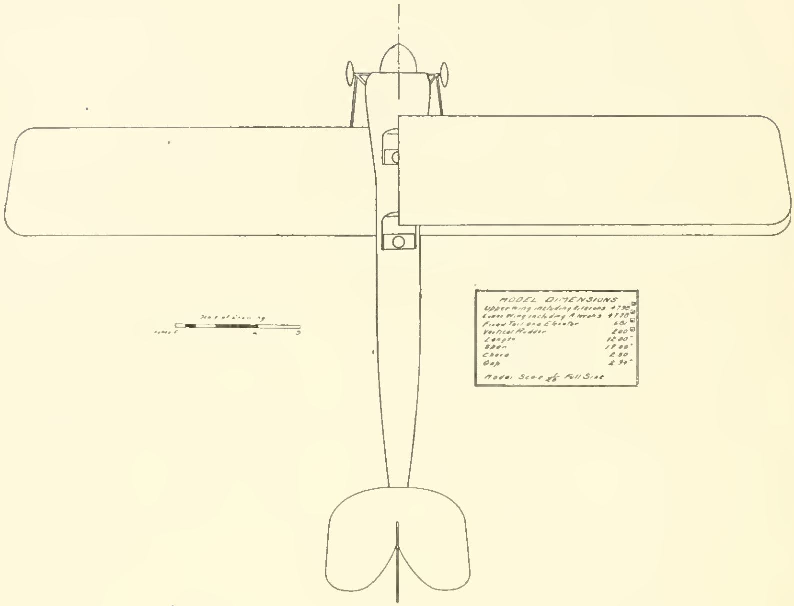

MODEL DIMENSIONS
Vertical Rudder
Length
Span
Chord
Gap
Full Size

FIG. 1A.

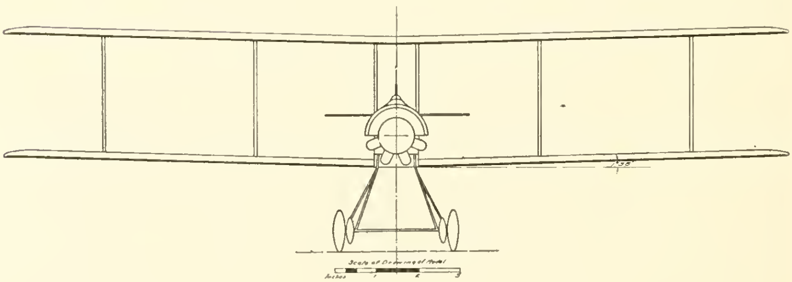


FIG. 1B.

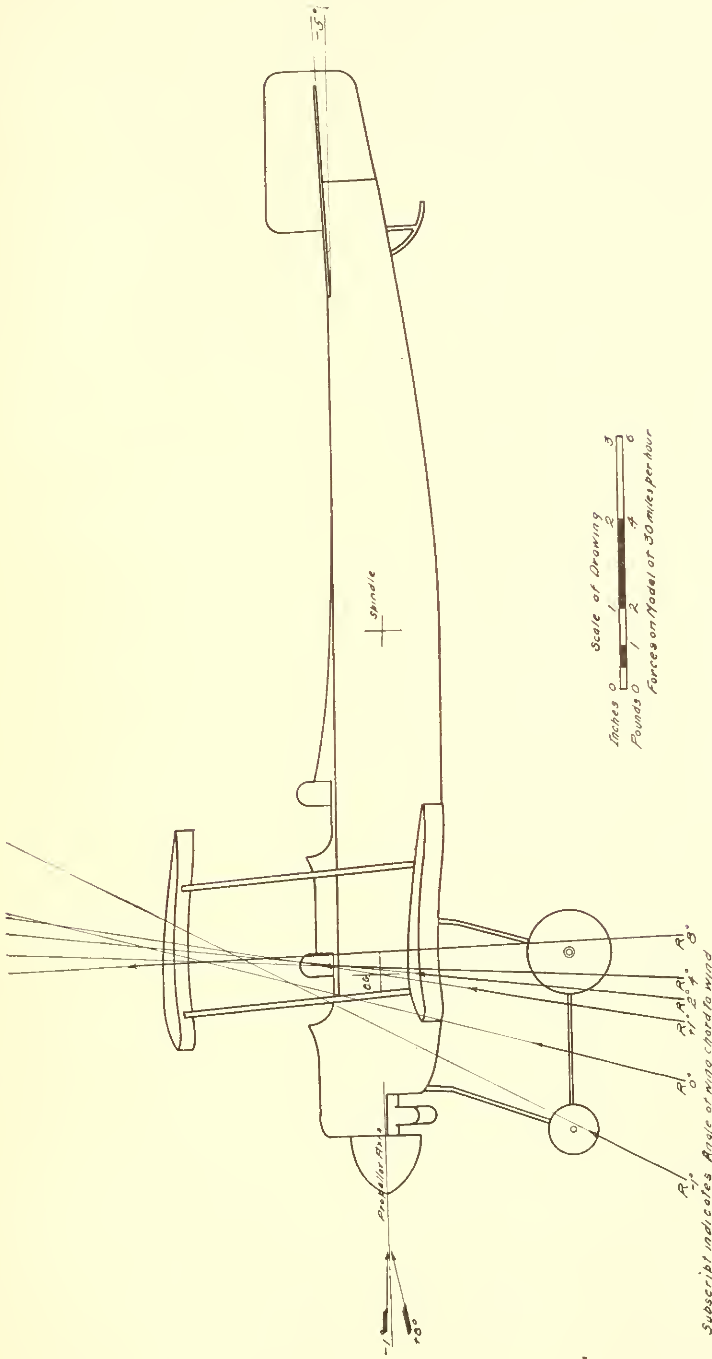


FIG. 1C. Resultant force vectors; tail —5°.

The lift and drift coefficients *Ky* and *Kx* were computed from the observed *L* and *D,* using such units that the coefficient is pounds force per square foot area per mile hour velocity. Curves of coefficients are given on figure 2, which also shows the ratio *L/D,* a measure of

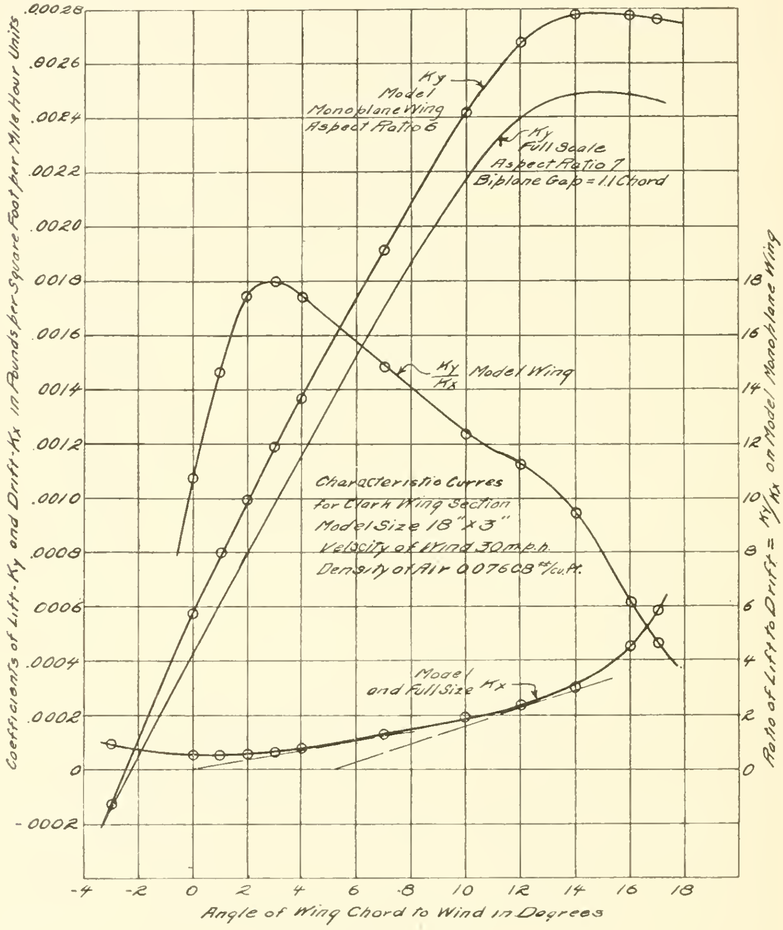
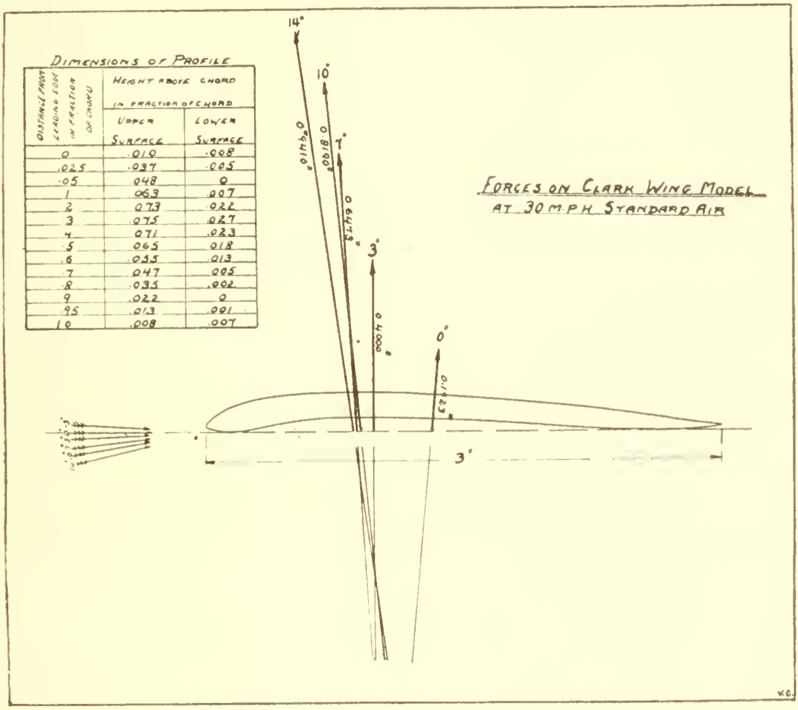


FIG. 2.—Wing coefficients.

the effectiveness of the wing. A maximum *L/D* ratio of 18 was found for an angle of attack of 3°. For a 41-foot wing at 70 miles per hour, it is believed that the lift coefficient is not greatly different, but that the drift coefficient at small angles is materially reduced. The effect is to increase the ratio *L/D.* Results of tests at the National Physical Laboratory (Tech. Rept. Adv. Comm. Aero., 1912-

13, p. 81) were applied to the L/D curve for our model to obtain an approximate curve of L/D to apply to the full-size wing. As a monoplane surface, we get a maximum value of L/D of about 20. The particular design is a biplane of aspect ratio 7. Well-known corrections for biplane interference loss and aspect ratio gain were applied to get a corrected curve for use in the design.



DIMENSIONS OF PROFILE

Distance from leading edge in fraction of chord	Height above chord in fraction of chord	
	Upper Surface	Lower Surface
0	.010	.008
.025	.037	.005
.05	.048	0
.1	.063	.007
.2	.073	.022
.3	.075	.027
.4	.071	.023
.5	.065	.018
.6	.055	.013
.7	.047	.005
.8	.035	.002
.9	.022	0
.95	.013	.001
1.0	.008	.007

FIG. 3.—Wing section dimensions and resultant force vectors.

The center of pressure for this wing is shown by figure 3, as well as the contour of the section. Center of pressure is defined as the intersection of the resultant force on the wing (represented as a vector) with the plane of the chord. It is seen that the wing section is unstable longitudinally at small angles. That is, if the wing heads down so that the angle of attack becomes $-3°$, the moment of the resultant force tends to turn it down still farther.

Applied to the aeroplane, it is necessary to balance and correct this tendency to dive by horizontal tail surfaces of proper size and attitude.

§5. LONGITUDINAL BALANCE

The complete model, using wings of the section described above and fitted with the tail shown in figure 1, was mounted in the wind tunnel on the balance with the wings vertical. A vertical spindle from the balance was driven into the side of the body at the point shown on figure 1. By swinging the model about the vertical axis passing through the spindle, the angle of the wind to the wing chord was varied from +20° to −8°. At each attitude the force across the wind or lift *L*, force down wind or drift *D*, and the pitching moment about the spindle were measured. The wind velocity was 30 miles per hour for all tests. The signs were taken so that an actual lift, actual drift, and a stalling moment are positive. Density of air is at 15° C., 776 mm. Hg., dry.

Test No. I was made with the horizontal tail surface making an angle of −2°.75 with the wing chord. That is to say, the rear edge of the tail was tilted up. Test No. II was a repetition but with the tail at −7°. Test No. III had the tail surface at −5°.

The lift and drift in pounds on the model at 30 miles per hour are given below, and on figure 4.

	Case I		Case II		Case III	
i	*L*	*D*	*L*	*D*	*L*	*D*
−4	−.049	+.1147	−.172	+.1363	−.115	+.128
−2	+.18	+.1011	+.035	+.1103	+.112	+.108
−1	+.32	.0988	+.143	.1047	+.240	+.104
0	+.454	.099	+.298	.1014	.360	.101
+1	.569	.1016	.437	.1011	.490	.102
+2	.703	.106	.572	.1028	.625	.105
4	.927	.121	.807	.1135	.872	.115
6	1.131	.143	....	....	....	...
8	1.32	.167	1.224	.1381	1.305	.153
10	1.484	.195	....	....	....	...
12	1.604	.236	1.537	.2213	1.568	.213
14	1.653	.313	....	....	....	...
16	....	...	1.640	.391	1.64	.370
18	1.606	.547	1.614	.509	1.58	.498

The lift and drift at first sight appear to differ for the three cases, but it will be observed that the maximum lift is 1.65, 1.64, and 1.64, and the minimum drift is .099, .101, and .101 for the three cases

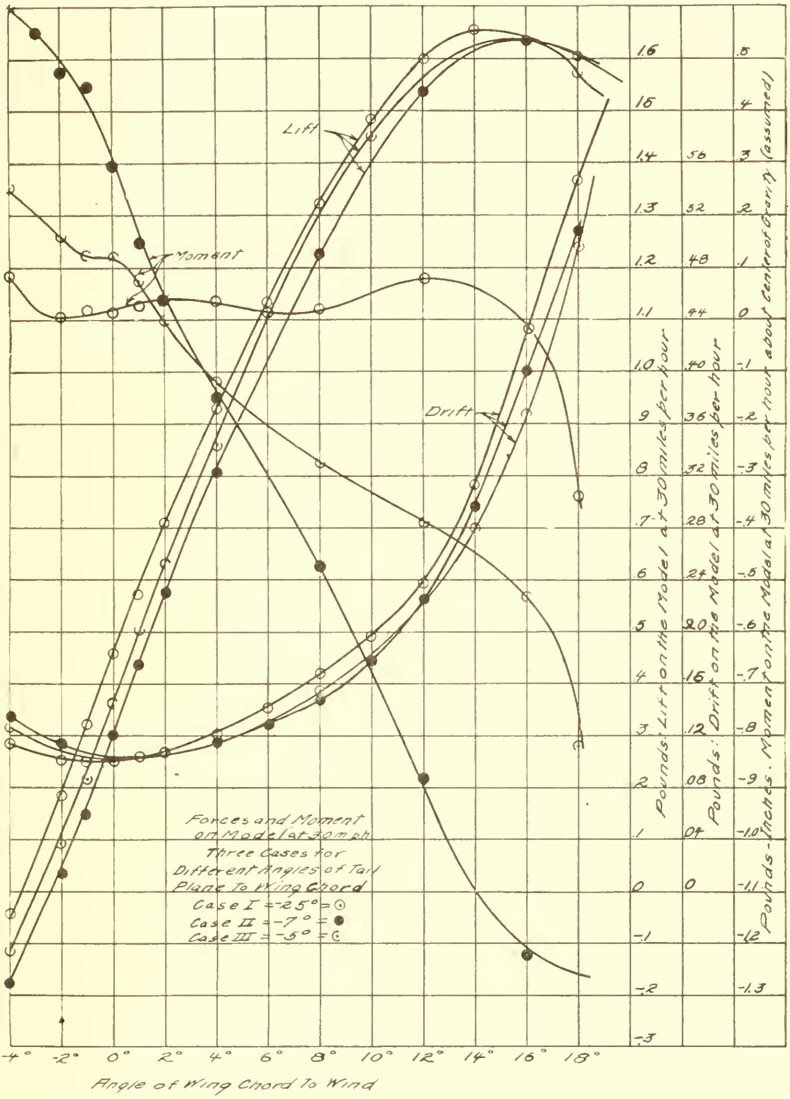


FIG. 4.—Curves L, D, and M for three tail settings.

respectively. The discrepancy is 1 per cent only and is about the precision of the measurements. The comparison is best brought out by eliminating reference to angle of attack as the effect of the change in tail angle appears to be mainly to move the curves of L and D, plotted on i, to the right or left.

Figure 5 shows the ratio L/D for the model for cases I, II, and III, plotted on L in pounds as abscissæ. For small values of L and angles of incidence between $-2°$ and $+2°$, corresponding in practice to high-flight velocity, the curves are practically identical. For angles

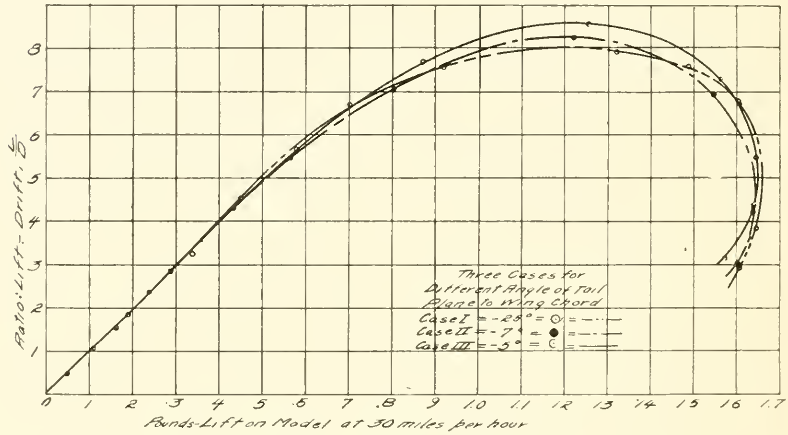


FIG. 5.—Curves of L/D plotted on L for three tail settings.

of incidence near 8°, the L/D ratio for case III is 8.6, while it is 8.2 for case II, and 8.0 for case I.

It appears, therefore, that changing the angle of tail surface has but slight effect on the lift and drift of the aeroplane. The actual aeroplane should have the same maximum and minimum speeds in any case since the maximum lift and minimum drift are about the same regardless of angle of tail surfaces.

The statical stability against longitudinal pitching is, however, very different for the three cases. Thus the pitching moments (observed about the spindle and converted to pitching moments about the assumed center of gravity) are as follows, in pounds-inches on the model at 30 miles per hour. Positive angles and positive moments are stalling angles and stalling moments respectively.

Pitching Moments, Pounds-Inches

i	Case I	Case II	Case III
−4	+.089	+ .599	+.26
−2	+.008	+ .473	+.16
−1	+.022	+ .454	+.12
0	+.016	+ .292	+.12
+1	+.030	+ .143	+.07
+2	+.037	+ .037	−.01
+4	+.039	− .159	−.12
6	+.016	...	..
8	+.023	− .476	−.28
10	...	...	..
12	+.086	− .884	−.39
14	...	...	..
16	−.013	−1.328	−.53
18	−.336	−1.378	−.82

Case I, with tail at −2°75, shows very small pitching moments and may be said to be neutral for ordinary angles of incidence. Thus if the aeroplane be flown at +2° incidence, in order to maintain balance at this attitude the pilot must impress a diving moment of −.037 pound-inch (on the model) to overcome the stalling moment +.037 given above. Then if the aeroplane be accidentally tilted up to +12° by a wind gust or other cause, in the new attitude the net pitching moment is still positive, and hence tends to tilt the machine still more. It is, therefore, unstable unless the pilot intervenes with the horizontal rudder.

For case II, tail at −7°, there is a strong righting moment always acting to prevent stalling or diving. The machine is very stable, in fact excessively so. For instance, flying at 2° incidence, the moment to be held by the pilot is very small. Suppose, however, he wishes to fly at +12° corresponding in the full-scale aeroplane to about 36 miles per hour. To maintain a balance at +12° incidence, he must exert a stalling moment by use of the horizontal rudder equal to about $\frac{.884}{12} \times (26)^3 \left(\frac{36}{30}\right)^2 = 1{,}970$ pounds-feet. The arm of the elevator is about 20 feet (distance aft of center of gravity), requiring a lift of 100 pounds on the elevator flaps. The elevator is able to exert this force if turned up about 10°. The elevator motion available for control in gusty air is thus largely used up in maintaining balance. The drift on this elevator flap may be over 20 pounds, making a waste of 3.5 propeller horse-power, or about 6 brake horse-power.

It is preferable to balance a machine at high speed by placing the center of gravity well forward. Then the pilot will have to carry his elevator turned up when flying at low speed. But at low speed, he is most in need of the full elevator motion for control of pitching. We, therefore, conclude that case II, with fixed stabilizer at $-7°$, is very much too stable or stiff longitudinally, and case I, with stabilizer at 2°.75, is not stable enough.

Case III, with stabilizer at $-5°$, appears to balance longitudinally at $+2°$ incidence, and at $+12°$ incidence to have (full size) a natural diving moment which could be held by a negative lift on the elevator of only about 44 pounds, corresponding to about 4° elevator angle. Consequently, it was decided to adopt the arrangement of case III for the subsequent stability investigation.

§6. VECTOR REPRESENTATION

A clearer conception of longitudinal balance is obtained by representing the resultant forces acting on the model as vectors. Thus, for case II, we observed on the balance the lift L and drift D. The resultant force acting was then of magnitude $R=\sqrt{L^2+D^2}$. This resultant force lay in a direction making an angle θ given by $\theta=\tan^{-1} L/D$. The line of action of this resultant was at a perpendicular distance from the spindle axis given by $d=M_s/R$, where M_s is the observed pitching moment about the spindle. The resultant force, R, is thus defined in magnitude, direction, and line of application, and may be represented graphically as a vector. In figure 1, the resultant force vectors for case III are drawn on the side elevation of the model. The model is considered to be fixed and the wind direction to change so that the angle of incidence varies from $-1°$ to $+8°$. The vectors are, therefore, drawn relative to the aeroplane.

The vector for 2° passes near the center of gravity. If it were desired to balance the machine at some other attitude, 6° for example, the center of gravity should be located at some point on the vector for 6°.

Note that on figure 1, for angles greater than 2°, the vectors pass to the rear of the center of gravity indicating diving moments and *vice versa*. Thus the machine is in stable equilibrium at 2°, and if deviated from this angle, righting moments are at once created which tend to restore the normal attitude.

Such stability is "inherent" in the design of the aeroplane and depends wholly on the location of the center of gravity and setting of the stabilizer. No automatic devices are required which may or may not function in an emergency. The inherent stability here shown is static only. Later we will investigate the effects of inertia and damping involved in dynamical inherent stability. However, dynamical stability is impossible unless there be statical stability, and before undertaking a study of the former property, we were obliged to provide a reasonable righting moment to oppose diving and stalling.

§7. PERFORMANCE CURVES

In the design of this aeroplane, the resistance, and hence the speed for given power, was estimated from tests on wings, body, struts, wires, etc., considered separately. The test results were corrected and expanded to full speed full size, using reasonable corrective factors. As is well known, the resistance of many parts does not increase so rapidly as the square of the speed, on account of skin friction. Making all allowances a speed of over 85 miles per hour was predicted for 110 brake horse-power.

If we use the lift and drift observed on the model $\left(\frac{1}{26} \text{ full size}\right)$ at 30 miles per hour and convert to full size by assuming the "law of squares," the performance is not quite so favorable and a maximum speed of but 75 miles per hour is indicated.

For a stability investigation we are little concerned with the exact speed, and for simplicity, the L and D from the wind tunnel test on the complete model of figure 1 are converted to full size by multiplying by the squares of speed and scale.

A total weight of 1600 pounds is assumed, corresponding to tanks half full. For any speed V the lift is a function of speed and attitude and must equal the weight W.

By the "law of squares"

$$\frac{\text{Force on Model}}{\text{Force on Aeroplane}} = \left(\frac{30}{26V}\right)^2,$$

hence:

$$V = \frac{30}{26}\sqrt{\frac{W}{L}},$$

where L is lift on model at 30 miles per hour.

For a series of values of L, corresponding to a series of attitudes or angles of incidence, the required speed V was computed. The head resistance of the aeroplane moving at these attitudes and with these speeds was computed from:

$$T = D\left(\frac{26V}{30}\right)^2,$$

where D is drift on model at 30 miles per hour, and T total thrust required.

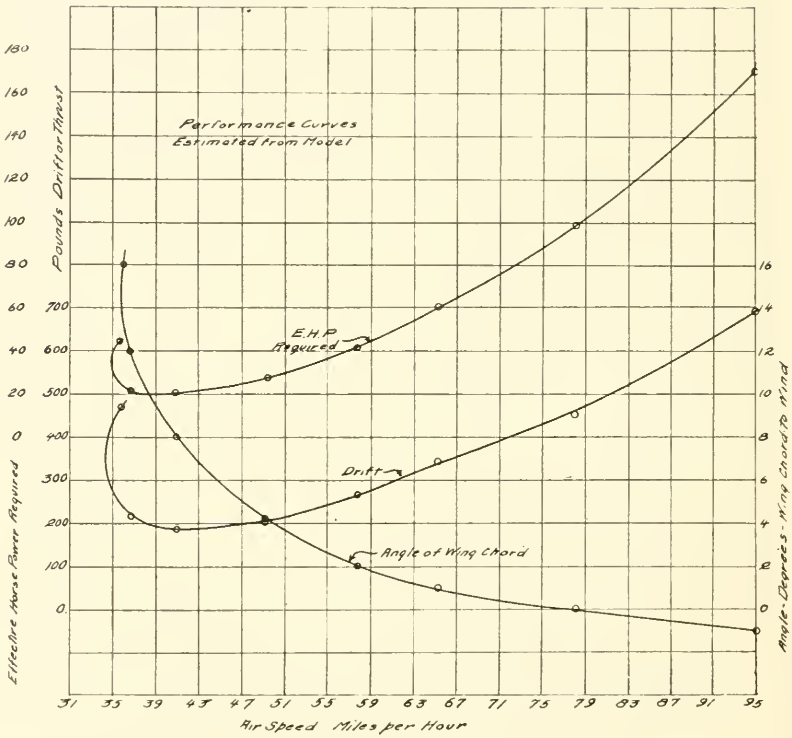


FIG. 6.—Characteristic performance curves.

The effective horse-power required, angle of wing chord to wind and thrust required are plotted as "characteristic performance curves" on figure 6.

§8. AXES AND NOTATION

We shall adopt a notation similar to Bairstow's for the study of dynamical stability. The normal attitude of the aeroplane is its position when in steady flight in a straight line. We select rectangular axes with origin at the center of gravity and fixed in the aeroplane and moving with it in space. In the normal attitude, the axis of x is tangent to the trajectory of the center of gravity with its positive direction toward the rear. The axis of z is normal to x and

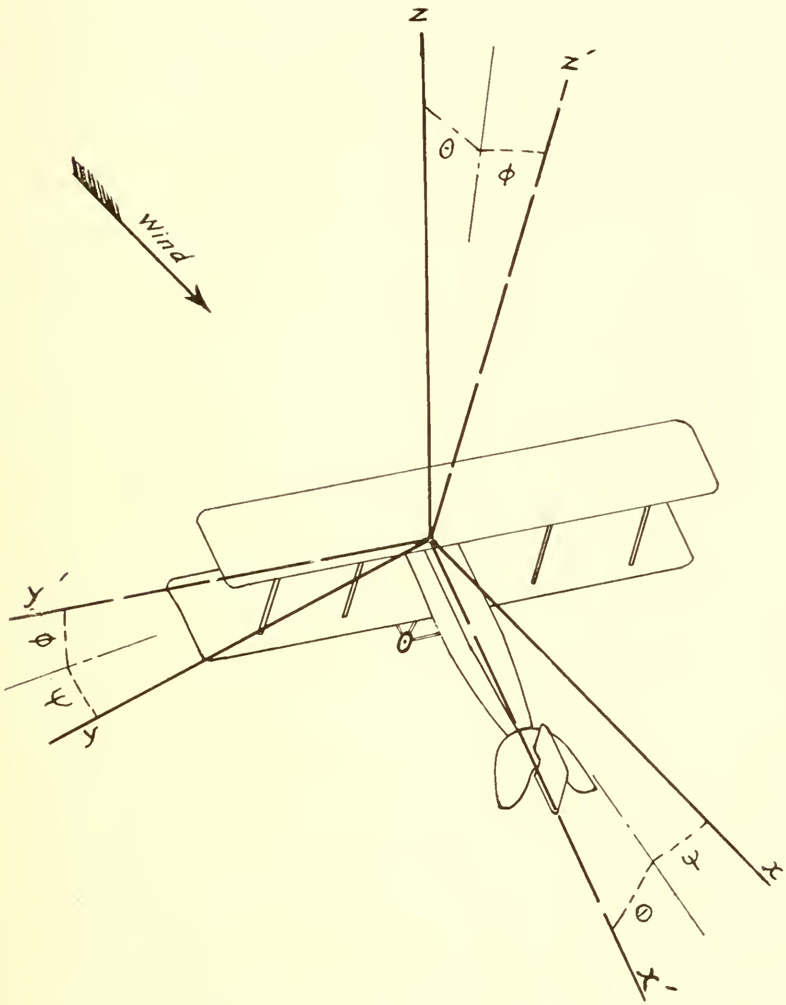


FIG. 7.—Coordinate axes, x, y, z.

y in the vertical plane, and the axis of y horizontal and directed to the left. The axes are shown in figure 7. As the aeroplane rolls, yaws, and pitches these axes move with it, so that z is no longer in the vertical plane of x, nor y horizontal.

Let the aerodynamical forces along the axes x, y, z be denoted by X, Y, Z and expressed in pounds force per unit mass.[1] The moments about these axes are L, M, N in pounds-feet per unit mass. Angular velocities about the axis are p, q, r in radians per second. Let angles of pitch, roll, and yaw away from the normal attitude be θ, ϕ, ψ in radians. Signs are positive in the directions xy, yz, and zx.

The radii of gyration about the axes x, y, z are K_A, K_B, K_C in feet. The mass of the aeroplane is m in slugs. The products of inertia are D, E, F. Two are zero for reasons of symmetry, and one is small in ordinary aeroplanes.

In normal flight in still air, the apparent wind blows in the positive direction of the axis of x. Let this velocity be produced by the forward velocity U of the aeroplane in normal flight. U is a negative number of feet per second.

Let small changes in velocity components along the axes x, y, z be u, v, w when any departure is made from the normal flying attitude.

In normal flight it is assumed that the power available maintains the aeroplane at such a speed that the weight is sustained and also that the normal attitude is that proper for the speed.

§9. EQUILIBRIUM CONDITIONS AND DYNAMICAL EQUATIONS OF MOTION

Let the inclination[2] of the flight path to the horizontal be θ_0. Since normal flight takes place in a straight line, $\psi_0=\phi_0=0$. There is no oscillation and $p_0=q_0=r_0=0$, and $L_0=N_0=0$.

If the propeller thrust T_0 be exerted in a line above or below the center of gravity h feet, then

$$M_0=-T_0h,$$
$$T_0=-g\sin\theta_0-X_0,$$
$$Z_0=g\cos\theta_0.$$

In this aeroplane $h=0$, and hence $M_0=0$.

If any accidental cause slightly disturbs the normal attitude of the aeroplane, the relative wind is no longer symmetrical and the aerodynamical forces and moments are X, Y, Z, L, M, N.

In general, the aerodynamical forces and moments caused by the deviation from "normal attitude" depend upon the relative motion of the aeroplane through the air, which motion is defined by U, u, v, w, p, q, r. Thus $X=f(U, u, v, w, p, q, r)$ where the form of the function f is not known; and five similar expressions for Y, Z, L, M, N.

[1] Unit mass is the slug of 32.2 pounds weight.

[2] Consider θ_0 positive for an upwardly inclined path as when climbing.

In the theory of small oscillations u, v, w, p, q, r are small by hypothesis and we may expand X by Maclaurin's theorem, neglecting squares and products of these small quantities. Hence,

$$X = X_0 + uX_u + vX_v + wX_w + pX_p + qX_q + rX_r,$$
$$Y = Y_0 + uY_u + vY_v + wY_w + pY_p + qY_q + rY_r,$$

and similar equations for Z, L, M, N.

Here X_u, X_r, etc., are the partial derivatives of X with respect to u, v, etc., and are the rates of change of X with u, v, etc. That is,

$$X_u = \frac{\partial X}{\partial U} = \frac{\partial X}{u}.$$

There are, therefore, 36 "resistance derivatives" involved which are constants for the aeroplane and depend upon the arrangement of surfaces and their presentation to the relative wind.

Fortunately, for reasons of symmetry, 18 of these derivatives vanish, for example: X_v, X_p, X_r. We then write:

$$X = X_0 + uX_u + wX_w + qX_q,$$
$$M = M_0 + uM_u + wM_w + qZ_q,$$
$$Z = Z_0 + uZ_u + wZ_w + qM_q,$$
$$Y = Y_0 + vY_v + pY_p + rY_r,$$
$$L = L_0 + vL_v + pL_p + rL_r,$$
$$N = N_0 + vN_v + pN_p + rN_r.$$

The above expressions are only approximate if u, v, w, etc., are not small.

The equations of motion for a rigid body having all degrees of freedom, are:

$$\frac{du}{dt} + wq - vr = X + T_0 + g\sin(\theta_0 + \theta),$$
$$\frac{dv}{dt} + (U+u)r - wp = Y - g\sin\phi,$$
$$\frac{dw}{dt} + vp - (U+u)q = Z - g\cos(\theta_0 + \theta),$$
$$\frac{dh_1}{dt} - rh_2 + qh_3 = mL,$$
$$\frac{dh_2}{dt} - ph_3 + rh_1 = mM + hT_0,$$
$$\frac{dh_3}{dt} - qh_1 + ph_2 = mN,$$

where

$$h_1 = pK_A^2m - qF - rE,$$
$$h_2 = qK_B^2m - rD - pF,$$
$$h_3 = rK_C^2m - pE - qD.$$

But the products of inertia (relative to moving axes fixed in the body) $D=F=0$, because the aeroplane is symmetrical about the xz plane. Substituting the above expressions for h_1, h_2, h_3, in the equations of motion, and neglecting products of small quantities, we have:

$$\frac{du}{dt}=X+T_0+g\sin(\theta_0+\theta),\quad K_2\frac{dp}{dt}-\frac{E}{m}\frac{dr}{dt}=L,$$

$$\frac{dv}{dt}+Ur=Y+g\sin\psi\sin(\theta_0+\theta)-g\sin\phi\cos(\theta_0+\theta),$$

$$K_B^2\frac{dq}{dt}=M+hT_0,$$

$$\frac{dw}{dt}-Uq=Z-g\cos(\theta_0+\theta),\quad K_C^2\frac{dr}{dt}-\frac{E}{m}\frac{dp}{dt}=N.$$

If we substitute for X, Y, etc., their values from the expansion in terms of the first powers of u, v, w, etc., and observing that from the conditions of equilibrium,

$$M_0+T_0h=T_0+X_0+g\sin\theta_0=Z_0-g\cos\theta_0=0,$$

we will have, making $\sin\phi=\phi$, $\sin\psi=\psi$, $\sin\theta=\theta$, and $\cos\theta=1$.

$$\frac{du}{dt}=uX_u+wX_w+qX_q+g\theta\cos\theta_0,$$

$$\frac{dw}{dt}=qU+uZ_u+wZ_w+qZ_q+g\theta\, sin\,\theta_0,$$

$$\frac{dv}{dt}=-rU+vY_v+pY_p+rY_r+g\psi\sin\theta_0-g\phi\cos\theta_0,$$

$$K_B^2\frac{dq}{dt}=uM_u+wM_w+qM_q,$$

$$K_A^2\frac{dp}{dt}-\frac{E}{M}\frac{dr}{dt}=vL_v+pL_p+rL_r,$$

$$K_C^2\frac{dr}{dt}-\frac{E}{M}\frac{dp}{dt}=vN_v+pN_p+rN_r.$$

We here assume T_0 a constant, or that there is no change of propeller thrust with small change in forward speed. With a motor in "free route," if the machine speeds up, the propeller tends to race or to speed up so that the slip shall be about constant, and hence the thrust is not materially changed. Since the forward speed $(U\pm u)$ is approximately equal to U, the thrust is approximately constant and equal to T_0.

We have also assumed that T_0 lies parallel to the axis of x. At very slow speed this is not exactly the case and T_0 has a small vertical component assisting in sustaining the weight of the aeroplane. At high speeds, T_0 is, however, usually parallel to x and the assumption that it always is so parallel is here made for simplicity. In any case T_0 is eliminated by the conditions of equilibrium.

In the present investigation the normal flight path is assumed horizontal, or $\theta_0=0$. The product of inertia E is small for ordinary aeroplanes with the heavy weights fairly symmetrical above and below the axis of x. In view of the probable insignificance of E and the fact that E cannot easily be determined for an aeroplane by simple experiments, it is here neglected. In the simplified form the equations of motion then are:

$$\frac{du}{dt}=uX_u+wX_w+qX_q+g\theta. \tag{1a}$$

$$\frac{dw}{dt}=qU+uZ_u+wZ_w+qZ_q, \tag{1a}$$

$$\frac{dv}{dt}=-g\phi-rU+vY_v+pY_p+rY_r. \tag{1b}$$

$$K_A\frac{dp}{dt}=vL_v+pL_p+rL_r. \tag{1b}$$

$$K_B^2\frac{dq}{dt}=uM_u+wM_w+qM_q. \tag{1a}$$

$$K_C^2\frac{dr}{dt}=vN_v+pN_p+rN_r. \tag{1b}$$

It is seen that equations (1a) involve only the longitudinal motion or motion in the plane of symmetry xz of the aeroplane, since p, r, v, and ϕ do not appear. Likewise, equations (1b) involve only the asymmetrical motion, lateral and directional, and do not contain θ, u, w, and q. The two sets may then be considered separately, the former on integration giving the "symmetrical motion" and the latter the "asymmetrical motion."

Since $\frac{d\theta}{dt}=q$, equations (1a) may be written in terms of three variables u, w, and θ and their first derivatives. The "resistance derivatives" X_u, X_v, X_q, etc., are constant coefficients. The three variables are each functions of the time, and the three equations at any instant of time must be satisfied by a concordant set of values of u, w, and θ. The equations are, therefore, simultaneous and are linear differential equations with constant coefficients.

Writing the operator D to indicate differentiation with regard to time or $\frac{d}{dt}$,

$$\left.\begin{aligned}(D-X_u)u-X_w w-(X_qD+g)\theta=0,\\ -Z_u u+(D-Z_w)w-(Z_q+U)D\theta=0,\\ -M_u u-M_w w+(K_B{}^2D^2-M_qD)\theta=0.\end{aligned}\right\} \tag{2a}$$

The right-hand members of these equations are no longer zero if any wind gusts are assumed.[1] The complementary function may be found by the well-known "operational method" by algebraic solution for D. (See: Wilson's "Advanced Calculus," p. 223.)

The physical condition that the three equations shall be simultaneous is expressed mathematically by equating to zero the determinant Δ formed by the coefficients of the variables u, w, and θ. Thus:

$$\Delta = \begin{vmatrix} D+X_u, & -X_w, & -(X_qD+q) \\ -Z_u, & D-Z_w, & -(Z_q+U)D \\ -M_u, & -M_w, & (K_B^2D^2-M_qD) \end{vmatrix} = 0.$$

Expanding the determinant we obtain:

$$A_1D^4+B_1D^3+C_1D^2+D_1D+E_1=0,$$

where for abbreviation:

$$A_1=K_B^2,$$

$$B_1=-(M_q+X_uK_B^2+Z_wK_B^2),$$

$$C_1=\begin{vmatrix} Z_w, & U+Z_q \\ M_w, & M_q \end{vmatrix} + \begin{vmatrix} X_u, & X_q \\ M_w, & M_q \end{vmatrix} + K_B^2 \begin{vmatrix} X_u, & X_w \\ Z_u, & Z_w \end{vmatrix}$$

$$D_1=-\begin{vmatrix} X_u, & X_w, & X_q \\ Z_u, & Z_w, & U+Z_q \\ M_u, & M_w, & M_q \end{vmatrix} - g \begin{vmatrix} M_u, & (-)\sin\theta_0 \\ M_w, & \cos\theta_0 \end{vmatrix},$$

$$E_1=-g\begin{vmatrix} X_u, & X_w, & \cos\theta_0 \\ Z_u, & Z_w, & \sin\theta_0 \\ M_u, & M_w, & 0 \end{vmatrix}.$$

The solution of the biquadratic Δ for D is of the form:

$$D=a,\ b,\ c, \text{ or } d,$$

$$\theta=K_1e^{at}+K_2e^{bt}+K_3e^{ct}+K_4e^{dt},$$

where K_1, K_2, K_3, K_4, K_5, ... K_{12} are constants determined by initial conditions. Solutions for u and w are similar.

The condition for stability of motion is that θ, u, and w shall diminish as time goes on. Hence, each of the roots of the biquadratic must be negative if real, or, if imaginary, must have its real part negative. This condition for stability may be applied without finding the constants K_1 to K_{12}, by solving only the biquadratic for a, b, c, d. Indeed, Bryan has shown that by use of Routh's discriminant the biquadratic need not be solved. The condition that a biquadratic equation have negative real roots or imaginary roots with real parts negative, is that A_1, B_1, C_1, D_1, E_1 and $B_1C_1D_1-A_1D_1^2-B_1^2E_1$ be each positive.

[1] Loc. cit., p. 1, §1, footnote 3.

In a similar manner the equations (1b) defining the asymmetric motion may be expressed as linear differential equations with constant coefficients.

Substitute $D^2\phi$ for $\frac{dp}{dt}$ and $D\phi$ for p.[1] Then:

$$(D-Y_v)v+(U-Y_r)r+(g-Y_pD)\phi=0,$$
$$-L_vv-L_rr+(K_A^2D^2-L_pD)\phi=0,$$
$$-N_vv+(K_C^2D-N_r)r-N_pD\phi=0,$$
$$\Delta_2=A_2D^4+B_2D^3+C_2D^2+D_2D+E_2=0,$$

where:

$$A_2=K_C^2K_A^2,$$
$$B_2=-Y_vK_C^2K_A^2-K_C^2L_p-N_rK_A^2,$$
$$C_2=-L_rN_p+N_rL_p+K_C^2L_pY_v+N_rY_vK_A^2+N_vUK_A^2-(L_vY_pK_C^2+N_vY_rK_A^2),$$
$$D_2=Y_v(L_rN_p-N_rL_p)+L_v(UN_p+gK_C^2)-UL_pN_v+(N_vY_rL_p-L_vY_rN_p+L_vY_pN_r-N_vY_pL_r),$$
$$E_2=g(N_vL_r-L_vN_r).$$

As before, the condition for stability is that the real roots and real parts of imaginary roots of the biquadratic be negative.

§10. CONVERSION TO MOVING AXES, LONGITUDINAL DATA

Horizontal flight at 0° incidence i of wing chord requires a speed of 112.5 feet per second, or about 77 miles per hour (see the characteristic performance curves). The normal attitude then has the axis of x parallel to the wing chord and horizontal. The axis z is vertical. For slow speed with an angle of incidence i of 12°, a speed of 54 feet per second, or about 37 miles per hour, must be maintained. In this case, the normal attitude has the axes x horizontal and z vertical, but the axes are entirely different from those used for the high-speed condition if they are considered with reference to the aeroplane. The axis of y is, however, the same in both cases.

[1] Since we consider only the small oscillations, ϕ and ψ are of the nature of infinitesimals, and hence compound vectorially as do p and r. Professor E. B. Wilson suggests the important implification of the treatment given by Bryan or Bairstow due to making $\frac{d\phi}{dt}=p$ and $\frac{d\psi}{dt}=r$. They used angular coordinates giving expressions for $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$ in terms of p and r and the angles which are initially cumbersome but ultimately reduce to the simple form here given.

The aeroplane may pitch about its normal attitude. At any instant the angle of pitch is the angle θ between the normal attitude axis of x and the new position of x. The axes, of course, pitch with the aeroplane. The axes are fixed by the equilibrium conditions and differ for each speed since each speed requires a different attitude.

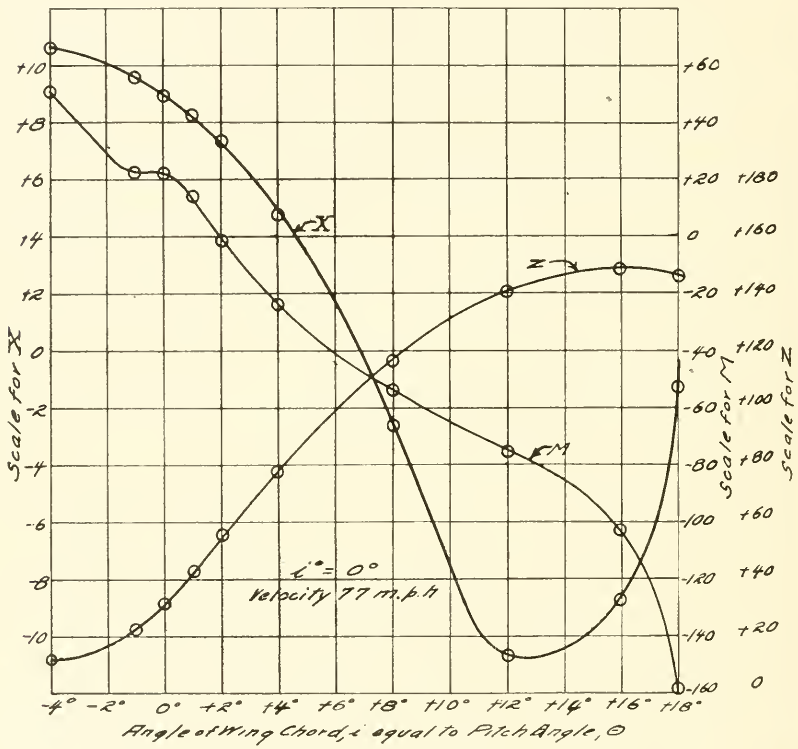


FIG. 8.—X, Z, and M for $i=0°$.

It was convenient to measure in the wind tunnel the lift and drift on the model referred to axes always vertical and horizontal. The corresponding forces along the moving axes x and z are readily obtained from:

$$Z' = L \cos\theta + D \sin\theta,$$
$$X' = D \cos\theta - L \sin\theta.$$

Here L and D are pounds on model, θ is angle of pitch, and Z' and X' are pounds force along the moving axes. X' and Z' are then converted to full-speed full scale as usual and divided by the mass m in slugs to obtain X and Z in pounds per unit mass on the full-size aeroplane at the proper speed.

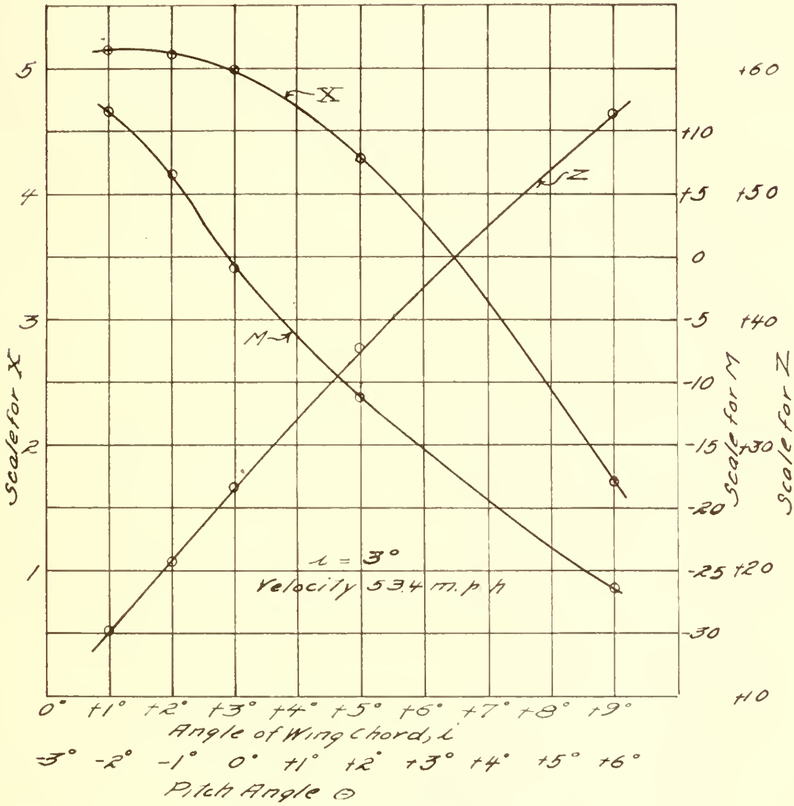


FIG. 9.—X, Z, and M for $i = 3°$.

The pitching moment full size is obtained from the observed model pitching moment about the center of gravity by an obvious manipulation. The moment is expressed in pounds-feet per unit mass and lettered M.

For this aeroplane we have, for example,

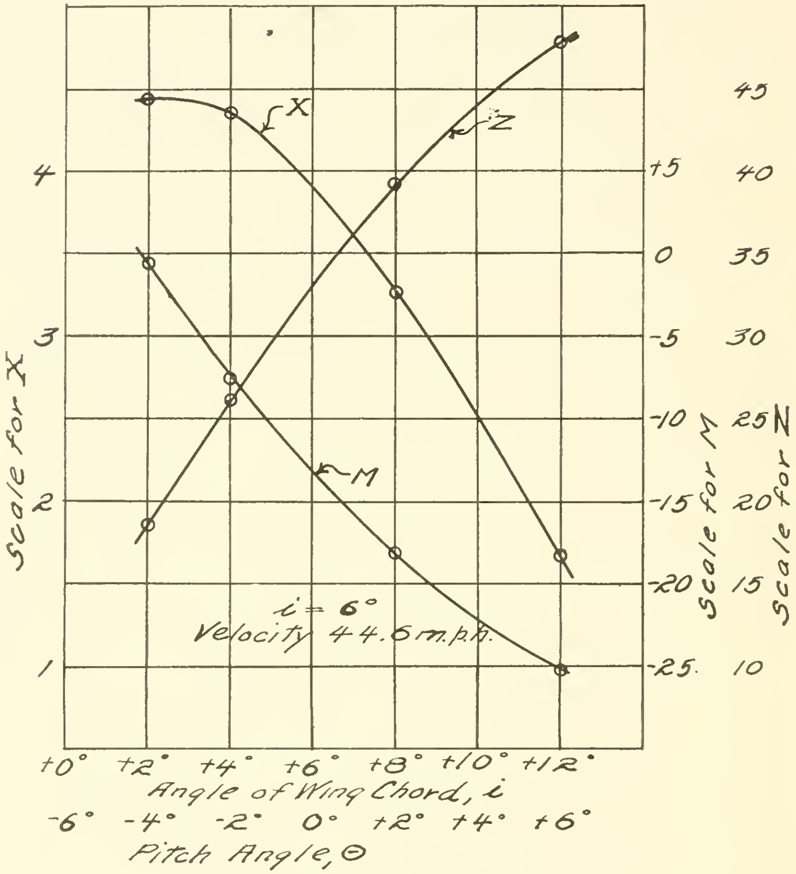


FIG. 10.—X, Z, and M for $i=6°$.

For this aeroplane we have, for example:

$m=50$ slugs, $\theta_0=0$,

$U=-112.5$ foot-seconds (high speed, 76.9 miles per hour),

$i=0$, normal attitude.

$i°$	$\theta°$	X	Z	M
−4	−4	+10.66	+ 11.00	+ 50.2
−1	−1	+ 9.62	+ 21.18	+ 23.2
0	0	+ 8.98	+ 32.06	+ 23.0
+1	+1	+ 8.28	+ 43.74	+ 13.5
2	2	+ 7.38	+ 56.00	− 1.93
4	4	+ 4.80	+ 78.26	− 23.2
8	8	− 2.58	117.00	− 54.0
12	12	−10.68	140.6	− 75.3
16	16	− 8.72	149.00	−102.3
18	18	− 1.25	147.00	−158.2

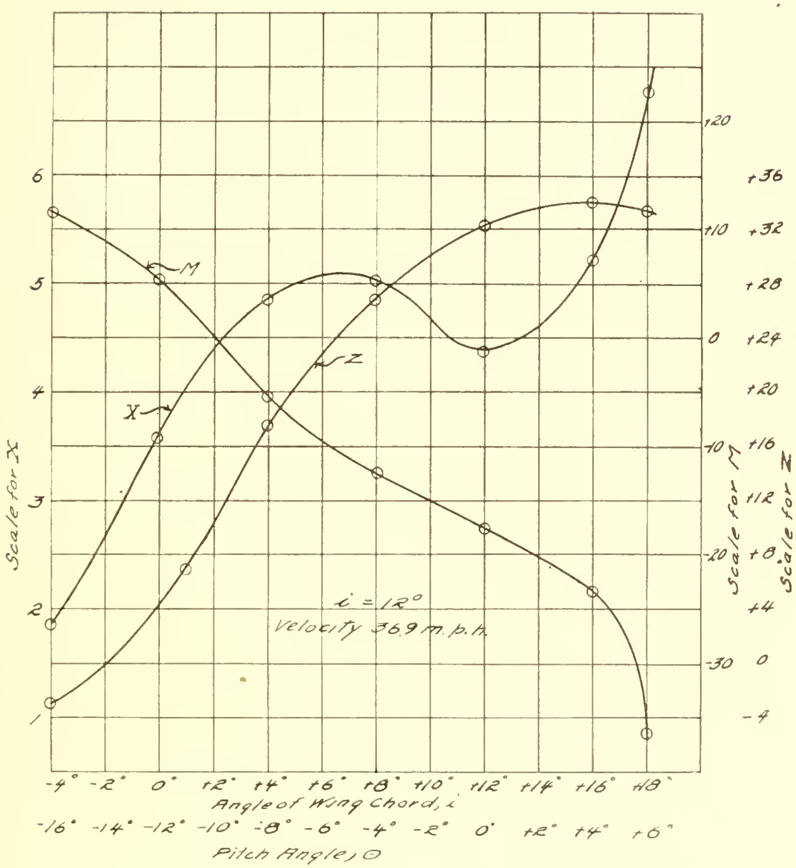


Fig. 11.—X, Z, and M for $i = 12°$.

$U = -54$ foot-seconds (low speed, 36.9 miles per hour).
$\theta_0 = 0$, $i = 12°$, normal attitude.

$i°$	$\theta°$	X	Z	M
−4	−16	+1.87	− 3.0	+11.6
0	−12	+3.58	+ 6.8	+ 5.30
+4	− 8	+4.84	+17.4	− 5.32
8	− 4	+5.02	+26.6	−12.5
12	0	+4.38	+32.3	−17.4
16	4	+5.24	+34.2	−23.6
18	6	+6.78	+33.4	−36.5

When $\theta = 0$, note that Z_0 should equal 32.2 or g, a check on the table.

Curves of X, Z, M for the four speed conditions are given on figures 8, 9, 10, and 11. These curves are not "faired," but drawn through the experimental points to show the consistency of the measurements and calculations.

§11. RESISTANCE DERIVATIVES, LONGITUDINAL

The longitudinal oscillations of the aeroplane are given by three equations of motion of §9, in which certain " resistance derivatives " are required.

The quantity X_u is the rate of change of X with change of forward speed u. Since X varies as the square of the speed, $X_0 = CU^2$ where C is some constant.

Then $\frac{\partial X}{\partial U} = 2CU = \frac{2X_0}{U} = X_u$ and $Z_u = \frac{2Z_0}{U} = \frac{2g}{U}$, so that these coefficients are readily calculated.

The derivatives X_w, Z_w, M_w represent the effect of a vertical component of velocity w. The vertical component of velocity w acts with the horizontal velocity U to cause the resultant wind to have an inclination to the horizontal

$$\Delta\theta = \tan^{-1}\frac{w}{U} = 57.3\frac{w}{U},$$

when $\Delta\theta$ is a small angle measured in degrees.

Hence

$$X_w = \frac{\Delta X}{w} = \frac{57.3}{U}\cdot\frac{\Delta X}{\Delta\theta}.$$

$$Z_w = \frac{57.3}{U}\cdot\frac{\Delta Z}{\Delta\theta}.$$

$$M_w = \frac{57.3}{U}\cdot\frac{\Delta M}{\Delta\theta}.$$

The method practically substitutes the slopes $\frac{\Delta X}{\Delta\theta}$, $\frac{\Delta Z}{\Delta\theta}$, $\frac{\Delta M}{\Delta\theta}$ of the tangents to curves of X, Z, M, at $\theta = 0$, for the actual curves. We have assumed $\Delta\theta$ small. If a curve be nearly a straight line, we may substitute the tangent for the curve without great error. Thus it may not always be necessary to assume $\Delta\theta$ *very* small. In fact, a range of from 5° to 8° is tolerable.

Since we assume $M_0 = 0$, the balance should be undisturbed by change of forward speed. Therefore, $M_u = 0$ in all cases.

Note that a positive value of M_w corresponds to a curve of pitching moments giving statical stability or a righting moment. If M_w is positive it does not necessarily follow that the aeroplane will be dynamically stable, but if M_w is negative, instability is of course certain. X_u should be negative to indicate increased resistance for increase of forward speed $-u$. For stability, Z_w should be large and negative, indicating increase lift for larger angles of incidence and *vice versa*. At stalling angles, Z_w tends to approach zero.

§12. DAMPING

The derivative M_q is the rate of change of pitching moment due to angular velocity, or rapidity of pitching q. For a pitch of velocity $\frac{d\theta}{dt}=q$, there is a moment of qM_q tending to resist such pitching. This is the damping due to the horizontal stabilizer, elevator flaps, body, and all parts forward and aft of the center of gravity. The pitching takes place about the center of gravity. The damping is increased by a large tail and a long body.

The damping of a surface should depend on the area of the surface, the moment arm of that surface, the linear velocity with which it swings through the air (which varies also as the moment arm), and with the velocity of advance. Thus: $qM_q \sim ql^4U$, where l is a linear dimension.

If we can measure M_q for the model at any wind speed, we may convert it to M_q for the full-scale aeroplane at its proper speed by multiplying by the fourth power of the scale and the ratio of aeroplane speed to wind speed. Naturally this is an approximate method, but it is the best available since full-scale tests for M_q are not practicable.

Similarly N_r and L_p may be obtained from model tests. These refer to the damping of a yaw and a roll respectively.

In order to measure M_q, N_r, and L_p a special oscillator was designed, shown in the photograph in figure 12. By setting the apparatus to oscillate in pitch, roll, or yaw the corresponding damping coefficients can be computed from the observed decrement. The photograph (pl. 1) shows the apparatus with model as used for pitching oscillations.

Let:

I = moment of inertia of all oscillating parts in slug foot units,

m' = mass of all oscillating parts in slugs,

M_0 = moment of air forces on model at rest,

M_s = moment of springs at rest,

$K\theta$ = additional moment of springs when deflected,

c = center of gravity of entire apparatus above pivot, feet,

θ = angle of pitch from normal attitude in radians,

$\mu_0 \frac{d\theta}{dt}$ = damping moment due to friction,

$\mu_w \frac{d\theta}{dt}$ = damping moment due to wind on apparatus,

$\mu_m \frac{d\theta}{dt}$ = damping moment due to wind on model,

$cm'\theta$ = static moment due to gravity.

The equation of motion then is:

$$I\frac{d^2\theta}{dt^2}+(\mu_0+\mu_w+\mu_m)\frac{d\theta}{dt}+(K-cm')\theta+M_0-M_s=0.$$

But $M_0=M_s$ by the initial condition of equilibrium. Let

$$\mu=\mu_0+\mu_w+\mu_m;$$

then

$$I\frac{d^2\theta}{dt^2}+\mu\frac{d\theta}{dt}+(K-cm')\theta=0.$$

The solution of this equation is well known to be:

$$\theta=Ce^{-\frac{\mu t}{2I}}\cos\left\{t\sqrt{(K-cm')\frac{1}{I}-\frac{\mu^2}{4I^2}}+a\right\},$$

where C and a are arbitrary constants. If time be counted when the amplitude of swing is a maximum, then $\cos\{\ \}=1$, and $\theta=\theta_0$, the initial displacement. Also if the number of beats be counted by observing the times for succeeding maxima, a plot of amplitude on time will have for its equation the simple form:

$$\theta=\theta_0 e^{-\frac{\mu t}{2I}}.$$

The coefficient μ is the logarithmic decrement of the oscillation and must be numerically positive to insure that the oscillation dies out with time.

The apparatus was fitted with a small reflecting prism by which a pencil of light was deflected toward a ground glass plate set in the roof of the tunnel. Nine lines spaced 0.2 inch were ruled on this plate. With the model at rest the beam of light was brought to a sharp focus on the line marked zero. By means of a trigger the observer started an oscillation of the model, and the spot of light was observed to oscillate across the scale. The time t was observed in which an oscillation was damped from an amplitude of 9 to an amplitude of 1, for example.

Then: $\log_e\frac{\theta_0}{\theta}=\frac{\mu}{2I}t=\log_e 9$, and knowing I and t, μ is calculated.

Preliminary tests showed that the same value of μ was obtained whether the timing stopped at $\theta=5$, 4, 3, 2, or 1.

Oscillation tests were made at five wind velocities varying from 5 to 35 miles per hour. The coefficient μ appeared to vary approximately as the first power of the velocity.

Similar tests were made with the model for no wind to determine μ_0, which may be said to be due almost wholly to friction and very slightly to the damping of apparatus and model moving through the air.

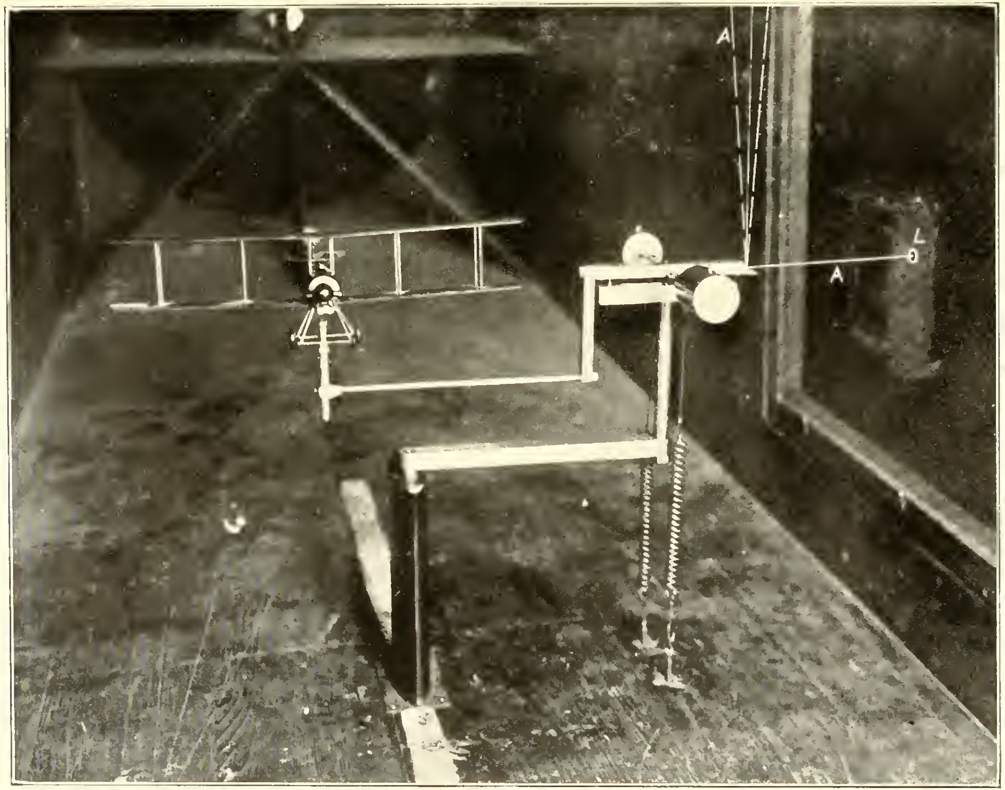


FIG. 12.—MODEL IN POSITION FOR PITCHING OSCILLATIONS ABOUT CENTER OF GRAVITY. L SPECTACLE LENS A, PENCIL OF LIGHT DEFLECTED TO SCALE ON ROOF

Likewise μ_w was obtained by oscillating the apparatus without model in winds from 5 to 35 miles per hour.

The coefficient μ_m has the dimensions $\rho l^3 V$, where ρ is density of air, l a linear dimension, and V the velocity of the wind. To convert μ_m to M_q for the full-size machine at full speed, multiply by the fourth power of 24, the scale, and by the ratio of full speed to model speed.

The model is mounted in such a manner that the axis of oscillation through the two steel pivot points passes through the assumed center of gravity location for the aeroplane. The actual center of gravity of the model is not considered.

Transverse arms carry counter weights by which the natural period may be adjusted. The springs insure that the motion shall be oscillatory. Knife-edged shackles bearing in notches in the transverse arms carry the pull of the springs. The springs are not calibrated as the calculation eliminates the spring coefficient.

Friction is kept small by careful design. All pivots are glass-hard tool-steel points bearing inside polished conical depressions of tool steel. A convenient period for observation is $\frac{1}{2}$ second. In still air, the apparatus will oscillate over 300 times before the amplitude is diminished to $\frac{1}{9}$ the initial displacement. The latter is about 3°.

Numerical results for the pitching oscillation follow:

§13. OSCILLATIONS IN PITCH

Inertia, model and apparatus = .03945
Inertia, apparatus = .03680

APPARATUS

Wind velocity, miles per hour	30	20	0
t, seconds	94.0	96.2	105
μ	.00172	.00168	.00154
μ_w (less zero)	.00018	.00014	0

APPARATUS AND MODEL, INCIDENCE OF WING, 0°

Velocity, miles per hour	35	30	24	18	8
t, seconds	15.5	17.5	21.0	26.5	50
μ	.0112	.00994	.00828	.00656	.00348
μ_0	.0015	.00154	.00154	.00154	.00154
μ_w	.0002	.0002	.00016	.00012	.00005
μ_m (net)	.00950	.00820	.00658	.00490	.00189

APPARATUS AND MODEL, INCIDENCE OF WING, 6°

Velocity	30	24
t	20	24
μ	.00870	.00725
μ_0	.00160	.00156
μ_w	.0002	.0002
μ_m (net)	.00690	.00550

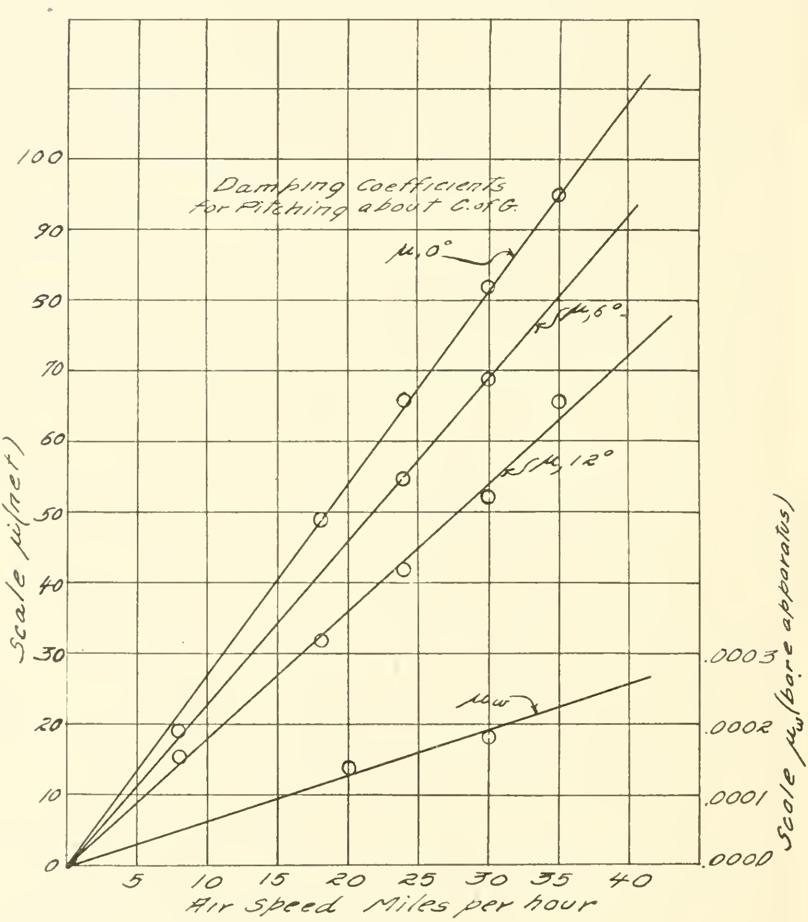


FIG. 13.—Curves for μ (net) and μ_w for oscillations in pitch.

APPARATUS AND MODEL, INCIDENCE OF WING, 12°

Velocity	35	30	24	18	8	0
t	23.5	25.0	29.0	35.5	55.5	112
μ	.0074	.00696	.0060	.0049	.00314	.00156
μ_0	.0016	.00156	.0016	.0016	.00156	.00156
μ_w	.0002	.0002	.0002	.0001	.00005	.00000
μ_m (net)	.0066	.0052	.0042	.0032	.00153	.00000

Values computed as above for μ_m, net, for the three cases are plotted in figure 13. The points appear to lie along straight lines in justification of the assumption that the damping coefficient varies as the first power of the velocity of flight. To convert to full-speed full-scale, we use the formula,

$$M_q = \frac{\mu_m}{m}(26)^4 \left[\frac{\text{Velocity aeroplane}}{\text{Velocity model}}\right],$$

$$\text{for } i=0°, \quad M_q=(-)192.0=1.71U,$$
$$i=6°, \quad M_q=(-)\ 93.7=1.43U,$$
$$i=12°, \quad M_q=(-)\ 60.5=1.12U.$$

The marked decrease in damping at slow speed must impair stability. For the Curtiss Tractor JN2, with a somewhat shorter tail, we found $M_q=1.32U$ at $i=2°$, and $M_q=1.66U$ at $i=15°.5$. Bairstow found for the Blériot, $M_q=1.84U$ at $i=6°$. We should expect greater damping to be shown there, since the horizontal tail surface is very large.

§14. LONGITUDINAL STABILITY, DYNAMICAL

We have now determined the resistance derivatives needed for the three equations of the longitudinal motion in the plane of symmetry with the exception of X_q and Z_q. From a consideration of various terms in the *criteria* for stability it is concluded that both X_q and Z_q enter into products which are small and relatively unimportant. They are consequently neglected.

The biquadratic has been calculated, following the formulæ given above, for several speeds and attitudes of flight. The results are summarized in the following table. The curves of figures 8, 9, 10, and 11 were used to obtain the resistance derivatives.

V miles per hour	76.9	53.4	44.6	36.9
U feet per sec..	− 112.5	− 78.2	− 65.3	− 54.0
Normal incidence	0°	3°	6°	12°
X_u	− .158	− .12	− .1194	− .162
X_w	+ .356	+ .249	+ .245	0
Z_u	− .57	− .823	− .985	− 1.19
Z_w	− 5.62	− 3.77	− 2.92	− 1.0
M_w	+ 3.2	+ 3.99	+ 2.25	+ 1.41
M_q	− 192.0	− 123.0[1]	− 93.7	− 60.5
A_1	21.6	21.6	21.6	21.6
B_1	317.0	207.0	159.3	85.1
C_1	1492.0	804.0	444.0	150.0
D_1	266.0	128.3	72.6	22.1
E_1	59.2	106.0	71.4	54.0
Routh's discr. ..	$+117\times 10^6$	16.4×10^6	3.2×10^6	$-.12\times 10^6$
m	50	50	50	50
Long period, sec.	34.7	17.6	15.8	10.56
Time to damp, 50%	8.1	11.0	13.1	....
Time to double.	....	....	....	24.7
Character......	Stable	Stable	Stable	Unstable

The coefficients of the biquadratic computed from the formulæ of §9 give for high speed

$$21.62D^4+317.0D^3+1492.0D^2+266.0D+59.2=0.$$

Each coefficient is positive and Routh's discriminant

$$B_1C_1D_1-A_1D_1^2-B_1^2E_1$$

is also positive and equal to 117×10^6. The motion is, therefore, stable. The aeroplane if set pitching will return in time to its normal attitude.

Bairstow has shown that, considering the usual values of the coefficients of the biquadratic, it may be factored approximately, giving:

$$\left(D^2+\frac{B_1}{A_1}D+\frac{C_1}{A_1}\right)\quad\left(D^2+\left[\frac{D_1}{C_1}-\frac{B_1E_1}{C_1^2}\right]D+\frac{E_1}{C_1}\right)=0.$$

The first factor reduces to:

$$D^2+14.75D+69.0=0,$$

$$D=-7.38\pm 3.83i \text{ where } i=\sqrt{-1}.$$

This is the well-known condition for a simple damped oscillation of period,

$$p=\frac{2\pi}{3.83}=1.64 \text{ seconds},$$

[1] By interpolation.

and damped to one-half amplitude in time,

$$t = \frac{0.69}{7.38} = 0.094 \text{ second.}$$

For most aeroplanes, this first factor corresponds to a short oscillation so heavily damped that it is of no importance. Indeed, it could not be observed on the actual aeroplane in flight.

The second factor, similarly, reduces to:

$$D^2 + .17D + .04 = 0,$$

$$D = -.085 \pm .181i,$$

$$p = \frac{2\pi}{.181} = 34.7 \text{ seconds,}$$

$$t = \frac{0.69}{.085} = 8.1 \text{ seconds.}$$

This is a longer oscillation but heavily damped. The period of 34.7 seconds for the motion is great, and at high speed this aeroplane if left to itself after an accidental longitudinal disturbance should follow an undulating path with rising and sinking of the center of gravity, together with pitching and periodic changing of forward speed. There is an oscillation in u, w, and θ. In 34.7 seconds, the aeroplane runs 3900 feet, which is the distance from crest to crest of the flight path. In one period the amplitude of the undulation is almost completely damped. It is unlikely that this motion would be uncomfortable to the pilot even if the initial disturbance due to a gust or other cause were severe.

At high speed, this aeroplane is very stable compared with other machines which have been tested. The natural period of the Curtiss JN2 is about 34 seconds, damped 50 per cent in 11 seconds, according to calculations made by us. A Blériot monoplane model tested by Bairstow had a period of pitching of 25 seconds, damped 50 per cent in 15 seconds.

There is no other published data of this character. It appears that great statical stability or large M_w will give a stiff machine with a rapid period. Such a machine, though very stable, may be so violent in its motion as to lead the pilot to pronounce it unstable. The design tested here appears to have as easy a period as the Curtiss and Blériot, both considered very satisfactory in flight, together with greater damping.

High speed and a long tail tend to damp the pitching. What we aim to secure—namely, steadiness in flight—may better be obtained by large damping factors rather than by strong righting moments (statical stability). It is well known that the French monoplane pilots demanded at one time a neutral aeroplane with no stability whatever against pitching, on the ground that "stable" aeroplanes were too violent in their motion in gusty air. Another disadvantage of excessive statical stability lies in the tendency of the machine to "take charge" and take a preferred attitude relative to the wind at a time when such a maneuver may embarrass the pilot, as when approaching a landing. However, it appears possible that a machine with the minimum of "statical" stability may be given the maximum of damping and so have a very slow period of pitching. The motion will be nearly dead beat.

This digression with regard to damping *vs.* "statical" stability applies with equal force to the rolling and yawing motions of the aeroplane to be considered under "lateral stability."

For low speed, 36.9 miles, similar calculations give for the longitudinal motion

$$21.6D^4+85.1D^3+149.8D^2+22.1D+54=0.$$

Routh's discriminant

$$B_1C_1D_1-A_1D_1^2-B_1^2E_1=-12\times10^4.\quad \text{Unstable.}$$

Short oscillation:

$$D^2+(B_1/A_1)D+C_1/A_1=D^2+3.9D+6.94=0,$$
$$D=-1.95\pm1.77i,$$
$$p=\frac{2\pi}{1.77}=3.58 \text{ seconds},$$
$$t=\frac{0.69}{1.95}=.36 \text{ second to damp 50 per cent.}\quad \text{Stable.}$$

Long oscillation:

$$D^2+(D_1/C_1-B_1E_1/C_1^2)D+E_1/C_1=D^2-.056D+.36=0,$$
$$D=+.028\pm.594i,$$
$$p=\frac{2\pi}{.594}=10.56 \text{ seconds},$$
$$t=\frac{0.69}{-.028}=-24.7 \text{ seconds},$$

or $+24.7$ seconds will double the initial amplitude. Unstable.

At this speed Routh's discriminant is negative, indicating that the motion is unstable. The instability is seen to appear when the real parts of the roots corresponding to the long oscillation become positive. The motion is rapid: only 11 seconds' period compared with 35 seconds at high speed, and any initial displacement will double itself in two periods. The damping of the motion has vanished and although the increase of amplitude is not so rapid that there is danger

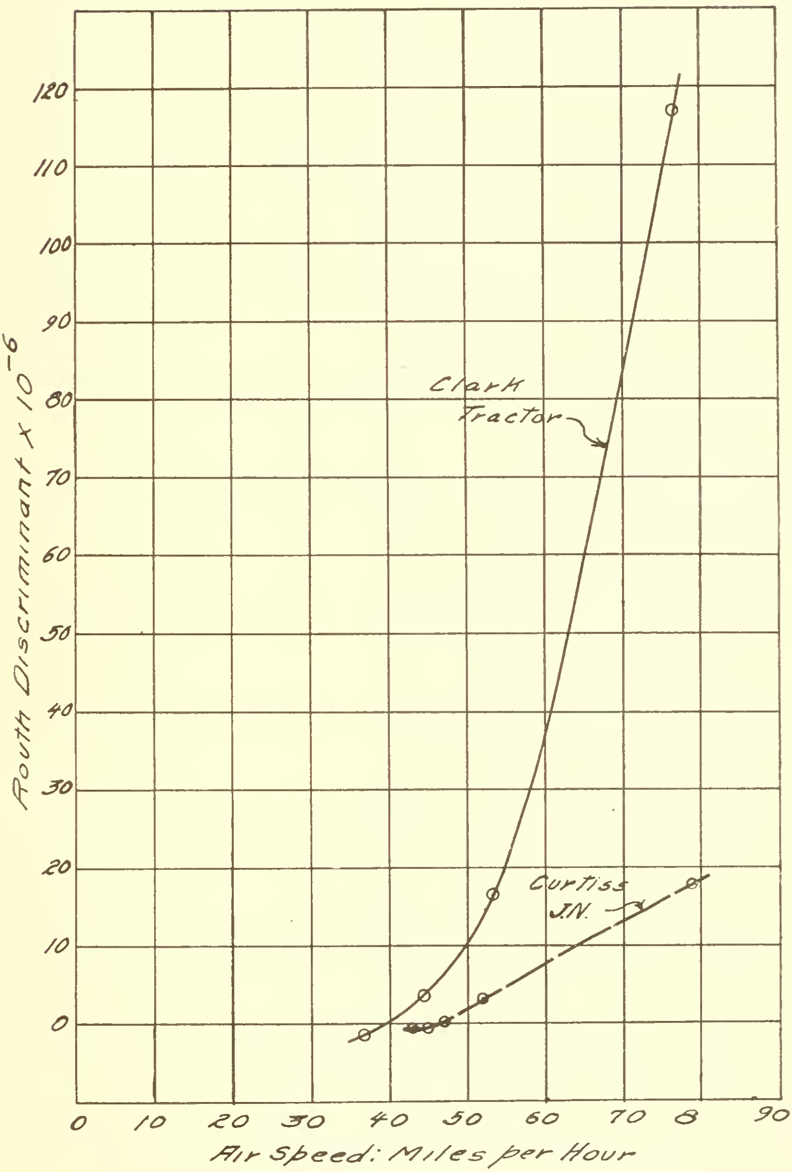


FIG. 14.—Routh's discriminant, variation with velocity.

of the pilot's losing control, yet it is clear that he cannot fly at this speed unless he is alert.

Taking Routh's discriminant as a measure of dynamical stability we have its value $+117 \times 10^6$ at high speed and -0.12×10^6 at low speed. Compared with the high-speed value, the latter is insignificant and we may conclude that the instability at low speeds is of relatively slight danger. Indeed, we may say that the aeroplane is stable at high speed and about neutral at low speed.

The progressive change in Routh's discriminant with speed is more clearly shown on figure 14. On the same plot, we give a similar curve for a Curtiss type tractor. The " critical velocity " for the Clark type is about 40 miles per hour and 47 miles per hour for the Curtiss type.

All aeroplanes of normal type are probably longitudinally stable at high speeds but lose this stability for all speeds below a certain critical speed where Routh's discriminant becomes zero or changes sign.

The examination of the longitudinal stability of the Blériot mentioned above applied only to high speed. The importance of investigating stability at low speeds has, it is believed, never before been shown.

The reason the stability of the longitudinal motion vanishes at a critical velocity must be found in the approximate factor representing the long oscillation.

$$D^2 + \left[\frac{D_1}{C_1} - \frac{B_1 E_1}{C_1^2} \right] D + \frac{E_1}{C_1} = 0.$$

Stability vanishes where $D_1/C_1 = E_1 B_1/C_1^2$, or where $D_1 C_1 = E_1 B_1$. In other words, stability is reduced as $E_1 B_1$ is made large or $D_1 C_1$ small. At high speed we have $266 \times 1492 > 59.2 \times 317$, but at low speed $22.1 \times 149.8 < 54 \times 85.1$. It appears that B_1 is smaller at low speeds, which is desired, but D_1 and C_1 are reduced to a greater degree, which is not desired.

The cause of the reduction in the magnitude of D_1 from 266 to 22.1 can be shown in the effect of change in resistance derivatives in:

$$D_1 = - \begin{vmatrix} X_u, & X_w, & X_q \\ Z_u, & Z_w, & U + Z_q \\ M_u, & M_w, & M_q \end{vmatrix} - g \begin{vmatrix} M_u, & \sin\theta_0 \\ M_w, & \cos\theta_0 \end{vmatrix} .$$

For $\theta_0 = 0$, $X_q = Z_q = M_u = 0$, we have

$$D_1 = -X_u Z_w M_q + X_u U M_w + Z_u X_w M_q .$$

The first term is reduced at low speed because Z_w is less than $\frac{1}{5}$ and M_q $\frac{1}{3}$ of their values at high speed. Since U and M_w are smaller, the second term is but $\frac{1}{6}$ of its high-speed value. The third term is unimportant.

From

$$C_1 = \begin{vmatrix} Z_w, & U \\ M_w, & M_q \end{vmatrix} + X_u M_q + K_B^2 (X_u Z_w - X_w Z_u)$$

we see by inspection that the principal reduction in C_1 at low speed is due to smaller values U, M_w, Z_w, and M_q which greatly reduce the terms $Z_w M_q$ and UM_w. These two terms are the principal numerical ones in the expression for C_1.

In general, $E_1 = -gZ_u M_w$ will increase in value due to increase in Z_u and M_w, but the effect on the motion is not great. On the other hand, $B_1 = -M_q - K_B^2 (X_u + Z_w)$ will drop rapidly for large angles of incidence due to drop in M_q and in Z_w. This is favorable to stability.

It is seen that the quantities U, Z_w, and M_q preponderate in the numerical values of the coefficients $D_1 C_1$ and $E_1 B_1$. For ordinary speeds, or speeds above the speed of minimum power, we have, approximately,

$$D_1 = -X_u (Z_w M_q - UM_w) + Z_u X_w M_q = -X_u (Z_w M_q - UM_w),$$
$$C_1 = (Z_w M_q - UM_w) + X_u M_q + K_B^2 (X_u Z_w - X_w Z_u) = (Z_w M_q - UM_w),$$
$$B_1 = -M_q - K_B^2 (X_u + Z_w) = -M_q - K_B^2 Z_w,$$
$$E_1 = -gZ_u M_w.$$

The condition for damped motion then becomes:

$$D_1 C_1 > E_1 B_1 \text{ or } (Z_w M_q - UM_w)^2 > -\frac{gZ_u}{X_u} M_w (M_q + K_B^2 Z_w),$$

where $\frac{Z_u}{X_u} = \frac{Z_0}{X_0}$ and M_w are nearly constant. Damping of the long oscillation is then favored by large values of Z_w, M_q, and U. That is, by light wing loading, large damping surfaces, and high velocity. As speed is reduced these quantities become smaller and the oscillation is less strongly damped.

For very low speeds, including those below the speed for minimum power, the value of Z_w nearly vanishes and M_q becomes small. Here the approximate expressions would be written,

$$D_1 = X_u UM_w + Z_u X_w M_q,$$
$$C_1 = -UM_w,$$
$$B_1 = -M_q - K_B^2 (X_u + Z_w),$$
$$E_1 = -gZ_u M_w,$$

and

$$-\left(\frac{X_0}{Z_0} M_w U + M_q X_w\right) > \frac{g}{U} \left(M_q + K_B^2 (X_u + Z_w)\right).$$

For very low speeds, the quantity X_w is often found to change sign; therefore, the two terms on the left may be of opposite sign and a large value for M_q diminishes D_1C_1 and increases B_1E_1. In a "stalling" attitude the aeroplane should have M_q small, M_w large, and, if possible, the radius of gyration in pitch K_B small.

The attitude of a "stalled" aeroplane is not ordinarily considered a "normal" attitude of flight, but, unfortunately, an aeroplane is frequently "stalled" by an inexperienced pilot. The longitudinal motion of an aeroplane if held in a "stall" would be, in general, unstable, but under favorable circumstances with Z_q, Z_w, K_B small and M_w large it is possible to have a stable motion. For example, in an extreme case with Z_w zero, if the aeroplane head up higher due to large X_w it slows down, loses lift and sinks. In sinking, M_w, if large, will head the machine down, speed will be gained on the dive and the resultant gain in lift causes the aeroplane to rise again. The oscillation will not increase in amplitude with time if the machine is able to respond quickly to the righting moment M_w. The damping M_q and radius of gyration K_B must not be too large. If M_q and K_B are too large, the machine is dynamically unstable by having $D_1C_1 < E_1B_1$.

The question of safe flight at a stalling attitude is complicated by the fact that the lateral controls become ineffective, but by manipulation of the power delivered by the motor, combined with skilful use of the rudder, an expert can land an aeroplane at surprisingly low speed.

The period is given by the imaginary part of the roots, or

$$p = \frac{2\pi}{\frac{1}{2}\sqrt{\frac{4E_1}{C_1} - \left(\frac{D_1C_1 - B_1E_1}{C_1^2}\right)^2}}.$$

Since $\left(\frac{D_1C_1 - B_1E_1}{C_1^2}\right)^2$ is usually small, we may write approximately,

$$p = 2\pi\sqrt{\frac{C_1}{E_1}},$$

$$= 2\pi\sqrt{\frac{Z_wM_q - UM_w}{-gZ_uM_w}}, \quad \text{but } Z_u = \frac{2g}{U},$$

then

$$p = \frac{\pi}{g}\sqrt{2U\left(U - \frac{Z_wM_q}{M_w}\right)}.$$

At low speed, U as well as Z_w and M_q are reduced and the period becomes short. A stiff machine with large M_w would have a rapid period. For given speed, if we make M_q large in order to provide heavy damping, care must be taken that M_w shall be small in order to secure a slow motion in pitch. It will be remembered that M_w is a measure of statical stability or "stiffness" and was mentioned as somewhat analogous to metacentric height for a ship.

By adjustment of Z_w, M_q, and M_w it appears possible to combine heavy damping with a fairly long period and so obtain great steadiness in normal flight.

§15. CONCLUSIONS (LONGITUDINAL DYNAMICAL STABILITY)

Stability calculations are of greater interest when they can be compared for different aeroplanes. At present, information is scanty but we may obtain by inference some general conclusions by comparing the Clark type aeroplane just described with a Curtiss type aeroplane previously tested by us.

The two aeroplanes are designed to have about the same performance. The principal difference at first sight is the greater wing area of the Clark—about 3.45 pounds per square foot against about 4.7 pounds per square foot for the Curtiss. In consequence of the lighter wing loading, the Clark type should have a steeper curve of Z giving Z_w large, which is favorable to stability.

The Clark aeroplane has a smaller horizontal tail area than the Curtiss, but the fixed part is inclined at $-5°$ to the wing chord against $-3.°5$ in the Curtiss. The Clark tail is only a trifle longer than the Curtiss and we may conclude that the pitching moment due to air pressure on the tail surfaces is about the same in the two machines. However, the Clark model uses a wing section on which the center of pressure motion for small angular changes is very slight. The Curtiss has a section described as R. A. F. 6 [1] in which this motion is considerable. For equal tail moments we may then expect M_w to be larger for the Clark machine. This is favorable to stability.

Due to the smaller tail, the damping of the pitching for the Clark model might be less than for the Curtiss. However, we find M_q at high speed -150 for the Curtiss against -192 for the Clark model. The increase must be due to the greater wing area of the latter since a calculation of the damping due to the tail alone gives a result less than one-half that observed for the whole machine.

The greater stability of the Clark model at high speeds is then due principally to greater values of Z_w and M_w. At low speeds, the resistance derivatives of these two aeroplanes are not greatly different. Both become very slightly unstable in their longitudinal motion.

[1] See Technical Report Advisory Committee for Aeronautics, 1912-13.

The following table facilitates comparison:

	Stable High Speed		Unstable Low Speed	
	Curtiss	Clark	Curtiss	Clark
i	1°	0°	14°	12°
X_u	− .128	− .158	− .223	− .162
X_w	+ .162	+ .356	− .132	0
Z_u	− .557	− .57	− .993	− 1.19
Z_w	− 3.95	− 5.62	− .555	− 1.0
M_w	+ 1.74	+ 3.2	+ 1.99	+ 1.41
M_q	−150.0	−192.0	−108.0	−60.5
K_B^2	34.0	21.6	34.0	21.6
A_1	34.0	21.6	34.0	21.6
B_1	289.0	317.0	134.0	85.1
C_1	834.0	1492.0	213.0	150.0
D_1	115.0	266.0	28.0	22.1
E_1	31.2	59.2	63.6	54.0
Routh's discr.	18×10^6	117×10^6	$-.37 \times 10^6$	$-.12 \times 10^6$
p sec.	34.0	34.7	11.5	10.6
t sec.	11.0	8.1	− 24.7	−24.7
U, ft.-sec.	−115.5	−112.5	− 64.8	−54.0

We may infer in general that:

1. Any ordinary aeroplane is likely to be unstable longitudinally below a certain critical speed.
2. Stability is improved by large wing area, *i. e.,* light load per square foot.
3. Stability is improved by large horizontal tail surfaces.
4. Stability is improved by high speed.
5. Stability is improved by great head resistance or a poor lift drift ratio.
6. Stability is improved by a small longitudinal moment of inertia.
7. Stability is improved by wings with slight center of pressure motion.[1]

There appears to be no reason to depart from the normal type of aeroplane in a search for longitudinal stability. A steady motion in flight is to be obtained by careful adjustment of surfaces in the ordinary type aeroplane, and the invention of freak types to accomplish great stability at the expense of speed or climb is to be discouraged.

Furthermore, the ordinary type of aeroplane may be made dynamically stable longitudinally without material sacrifice of desirable

[1] For a biplane combination giving a stationary center of pressure without material loss in other desirable features, see "Stable Biplane Arrangements," by J. C. Hunsaker, *Engineering,* London, Jan. 7 and 14, 1916.

flying qualities, such as ease of control. In this connection it is important not to give too great statical stability. Safety in flight may well depend more upon ease of control than upon stability. The almost universal prejudice among accomplished flyers against so-called "stable aeroplanes" appears to have a rational foundation.

PART II. LATERAL MOTION

§1. LATERAL OR ASYMMETRICAL TESTS

When the aeroplane is yawed to right or left of its course through an angle of yaw ψ, the wind blows through the wings obliquely and gives rise to a lateral force Y at right angles to the longitudinal axis x of the aeroplane, a rolling moment L tending to roll the aeroplane about the x axis, and a yawing moment N tending to yaw the machine about the z axis.

To measure the force Y and moments L and N as the aeroplane yaws, the model was mounted in the wind tunnel and held at various angles of yaw to the direction of the wind. At each position measurements were made from which the component forces X, Y, Z and moments L, M, N could be calculated.

The details of the method are given in the Technical Report of the Advisory Committee for Aeronautics, 1912-13, p. 128, where a description is found of the special apparatus required.

Briefly stated, the balance is arranged to measure the moments of the air forces about axes parallel to those axes used for calculation, whose origin is at the center of gravity of the aeroplane. A yawing moment is measured about a vertical axis passing through the main pivot of the balance. The moments of the drift and cross-wind forces are measured about horizontal axes parallel and at right angles to the tunnel axis and passing through the same point. In order completely to determine all forces and moments, a special fitting is provided on which three more measurements may be made. This moment device measures the pitching and rolling moments about horizontal axes passing through the pivot of the attachment. In addition, the total lift or vertical force is measured on the balance. We then have five moment observations and one force observation, as follows:

V_F, measured on vertical force lever (a lift),
M_Z, measured on torsion wire (a yawing moment),
V_P, pitching moment about a high point o_1,
V_R, rolling moment about a high point o_1,
M_D, moment of drift force about a low point o_2,
M_C, moment of cross-wind force about o_2.

We first reduce to the origin o_1 about which V_P and V_R are measured, which is l inches vertically above o_2.

Denote by primes forces and moments in pounds and pound-inches on the model for 30 miles per hour wind velocity referred to axes through the point o_2. Then:

$$L' = V_R \cos\theta - M_Z \sin\theta,$$
$$M' = V_P,$$
$$N' = V_R \sin\theta + M_Z \cos\theta,$$
$$X' = -V_F \sin\theta + \{M_D \cos\psi - M_C \sin\psi - V_P\} \frac{\cos\theta}{l},$$
$$Y' = \frac{V_R - M_C \cos\psi - M_D \sin\psi}{l},$$
$$Z' = V_F \cos\theta + \{M_D \cos\psi - M_C \sin\psi - V_P\} \frac{\sin\theta}{l}.$$

If the center of gravity of the aeroplane (model) be arranged to have the y coordinate zero, and its x and z coordinates a and b (in inches) referred to o_1, we have for the axes passing through the center of gravity:

$$X_1 = X',$$
$$Y_1 = Y',$$
$$Z_1 = Z',$$
$$L_1 = L' + cY',$$
$$M_1 = M' - cX' + aZ',$$
$$N_1 = N' - aY',$$

where X_1, Y_1, Z_1, L_1, M_1, N_1 are the quantities expressed in pounds and inch-pounds on the model at 30 miles per hour. Converting to full-speed full-scale and to units of pounds and pounds-feet per unit mass, we obtain the required X, Y, Z, L, M, N.

The model was first set at an angle of wing chord to wind of 0° corresponding to high speed. Measurements were then made as above for angles of yaw of ±25°, ±15°, ±10°, ±5°, 0°, keeping the incidence constant. In reducing the observations, values for left- and right-hand angles of yaw were averaged to eliminate errors due to lack of symmetry in the model. In the first test the angle of pitch θ is zero, and the axis of x horizontal. The test was repeated with the model at angles of incidence of 6° and 12°, corresponding to the intermediate and slow speed conditions. Here, again, θ in the formulæ of reduction is zero, since each new axis of x is also horizontal.

It is apparent that the labor involved in the complete solution for X, Y, Z, etc., is considerable and, unfortunately, the method requires the use of formulæ in which the difference between products of observed quantities is involved. Naturally, the precision of the result is poor when we are left with a small difference between large quantities.

The measurements V_F, M_Z, V_P, V_R, M_D, M_C are probably correct within 2 per cent. L involves no difference and may be taken as equally precise.

Since $N_1 = N' - aY'$ we may make the distance a very small in setting up the apparatus and so keep the precision of N about 2 per cent.

From

$$Y' = \frac{V_R - M_C \cos\psi - M_D \sin\psi}{l},$$

we note that $(M_C \cos\psi + M_D \sin\psi)$ is from three to five times as large as V_R. The precision of Y should then be between 2 and 6 per cent.

From similar reasoning, we may expect Z and X to be precise within 10 per cent, but in special cases, where we must take the difference of quantities of nearly equal magnitude, the precision is not so good.

The quantity M is a small moment which should be nearly zero if the aeroplane is balanced properly. Obviously, no estimate of the precision of M as a per cent can be given in such a case. Where M is large, as in the 12° condition, the measurement is precise to about 10 per cent.

Fortunately, for a study of lateral stability, we are concerned with Y, L, and N only, and these quantities are determined with fair precision.

The values computed for X, Z, and M for zero yaw may be compared with X, Z, M, determined independently in the tests on lift and drift discussed in Part I. The latter are probably precise within 2 per cent. Consequently the computed X, Z, and M obtained from the asymmetrical tests have been adjusted to make them agree with X, Z, M obtained from the symmetrical tests.

The change of X, Z, and M with ψ is not important, and X, Z, and M are not used in the theory of asymmetrical or lateral stability. Since by our equilibrium conditions, the pitching moment M_0 must be zero for normal flight, we must assume that the pilot makes M_0 zero by slight adjustment of his elevator flaps. In the tables below, the small value of M_0 observed when the angle of yaw ψ is zero has been subtracted from the observed M for each angle of yaw. This adjustment is required to give longitudinal equilibrium to the aeroplane when in its normal attitude.

The following tables summarize the data upon which the subsequent calculations are based:

High-speed attitude, $i=0°$, $l=26$ inches, $c=6.37$ inches, $a=-2.41$ inches.

OBSERVED

ψ	V_R	V_P	M_Z	M_D	M_C	V_F
0	0	1.74	0	4.25	0	.463
5	.284	1.74	.0333	4.41	.511	.453
10	.518	1.72	.0474	4.52	1.011	.450
15	.764	1.61	.0676	4.73	1.566	.445
25	1.222	1.52	.0851	5.43	2.650	.409

CALCULATED

ψ	X	Z	M	Y	L	N
0	9.00	32.2	0	0	0	0
5	9.99	31.5	+ 1.92	− 2.06	25.9	− 4.42
10	9.72	31.2	+ 1.92	− 4.31	40.2	−13.45
15	9.66	30.8	−15.37	− 6.74	54.0	−22.1
25	8.69	28.3	− 3.84	−12.23	66.9	−47.4

Intermediate-speed attitude, $i=6°$, $l=26$ inches, $c=6.67$ inches, $a=-2.06$ inches.

OBSERVED

ψ	V_R	V_P	M_Z	M_D	M_C	V_F
0	0	3.84	0	6.26	0	1.061
5	.416	3.80	.0041	6.39	.309	1.046
10	.776	3.72	.0090	6.50	.633	1.021
15	1.118	3.22	.0054	6.62	1.173	.993
25	1.455	2.69	−.0128	7.11	2.11	.891

CALCULATED

ψ	X	Z	M	Y	L	N
0	3.89	32.2	0	0	0	0
5	4.03	32.1	− 2.6	− .516	19.55	− 2.04
10	4.07	31.0	− 4.6	− 1.124	34.1	− 4.43
15	4.43	30.3	−37.6	− 1.99	43.8	− 8.54
25	4.40	27.2	−59.0	− 3.97	36.9	−18.6

Low-speed attitude, $i=12°$, $l=26$ inches,
$c=6.91$ inches, $a=-1.71$ inch.

Observed

ψ	V_R	V_P	M_Z	M_D	M_C	V_F
0	0	3.73	0	9.395	0	1.483
5	.348	3.71	−.0195	9.53	.094	1.464
10	.718	3.67	−.0541	9.61	.259	1.441
15	1.059	3.52	−.0847	9.68	.464	1.402
25	1.705	3.27	−.1455	9.86	.817	1.270

Calculated

ψ	X	Z	M	Y	L	N
0	4.4	32.2	0	0	0	0
5	4.5	31.7	0	− .45	8.65	− 2.53
10	4.46	31.2	0	− .95	17.6	− 5.95
15	4.43	29.8	− 3.5	− 1.51	24.7	− 9.35
25	4.14	27.1	0	− 2.45	37.8	−15.85

The variation, with angle of yaw, ψ, of the rolling moment L, yawing moment N, and lateral force Y, are shown by the curves of figure 15. In its symmetrical position, the aeroplane has no tendency to roll, yaw, or slide slip, and L_0, N_0, and Y_0 are zero, as stated in connection with the discussion of equilibrium conditions.

As the aeroplane yaws from its course, the plane of symmetry swings through an angle ψ, measured positive to the pilot's right hand. The momentum tends to carry the center of gravity forward in its original direction of motion. As a result, the apparent wind seems to strike the left cheek of the pilot. The curves of N show that, if this aeroplane yaw to the right, a negative yawing moment is produced which tends to turn the aeroplane to the left and hence to put it back on its course. The aeroplane is hence "directionally" stable, having a preponderance of fin surface behind the center of gravity, and the pilot need not use his rudder to stop the yaw. Numerically, we see that for a yaw of 10° at high speed, the value of N is -13.5 units, or about 670 pounds-feet. For a perfectly neutral aeroplane, to produce an equal yawing moment the pilot must exert a force of about 34 pounds on a vertical rudder 20 feet to the rear of the center of gravity.

When flying straight ahead, if the direction of the wind suddenly shifts so as to bring the apparent wind 10° to the left of the fore and aft axis of the aeroplane, the aeroplane tends to head over into the wind. An excessive amount of "directional" stability, indicated by a steep curve of yawing moments N, may cause the aeroplane to be unmanageable in gusty air. It may "take charge" and, due to excessive "weather helm," be difficult to keep on any desired course.

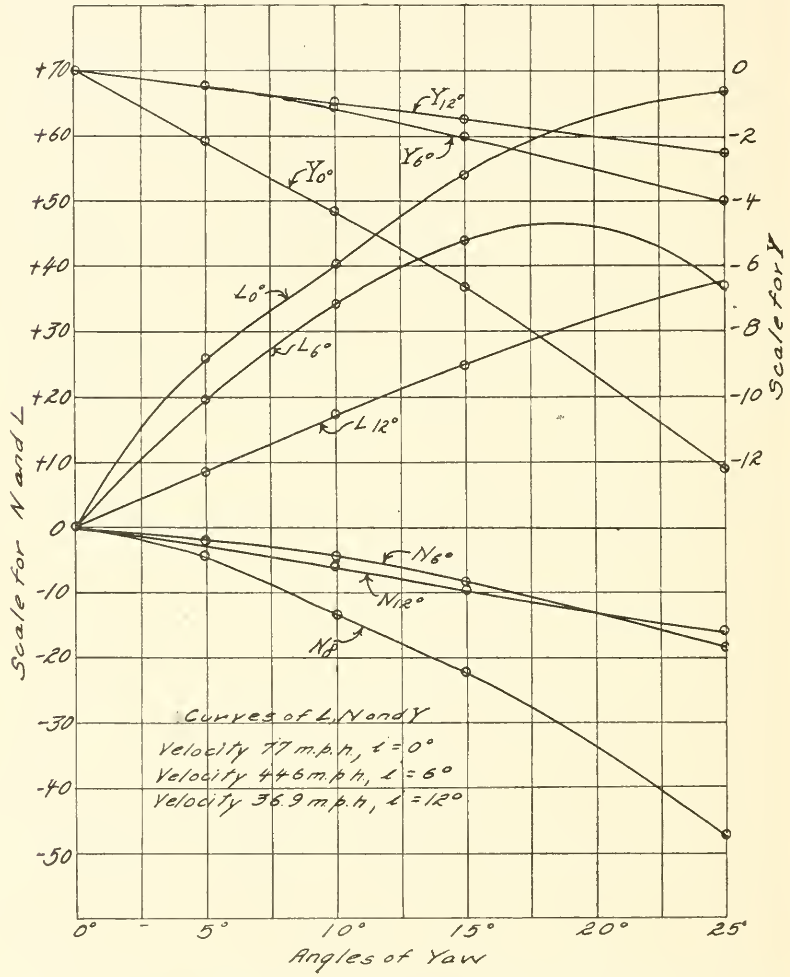


FIG. 15.—Curves of lateral force, rolling moment, and yawing moment, as angle of yaw changes.

It will be shown later that the so-called "directional" stability is not only undesirable in gusty air, but is the determining factor in "spiral instability." Indeed, "directional stability" is very nearly incompatible with inherent dynamical stability in roll, yaw, and side slip considered together.

If the aeroplane yaw to the right, it is practically starting off on a turn to the right. As is well known, to make such a turn safely an aeroplane should be "banked" to such an angle of roll that the centrifugal force, acting to the left, is about balanced by the horizontal component of the normal force Z acting to the right. In other words, the bank proper to a right turn requires a positive angle of roll ϕ given by a positive rolling moment L. The curves of L in the figure show that for this aeroplane the natural rolling or banking moments are positive for a positive yaw, and hence tend to bank the aeroplane suitably for the turn. This property is extremely valuable in preventing capsizing.

As in the case of the yawing moments, an excessive amount of natural banking may be uncomfortable, especially in gusty air. Thus, if the wind shifts to the left, the relative angle of yaw is positive, the aeroplane tends to turn to the left due to its "directional" stability and to bank for a turn to the right due to the natural banking or rolling moment L. The result may be to throw the aeroplane about in a somewhat violent manner, or it may capsize. This motion is discussed later under the heading "Dutch roll."

Large banking moments L can be given by vertical fin surface above the center of gravity, by a dihedral angle upwards or a "retreat" or sweep back of the wings. All these arrangements are probably equivalent and, though tending to give a stable motion in still air, tend toward violence in gusty air.

The model under test has, as is shown by the drawings, a dihedral angle upwards of the wings made by raising each wing tip 1.6°. This amount of dihedral has been found in practice to be not excessive on ordinary aeroplanes.

The curves of lateral force Y are negative for a positive yaw. This means that if the aeroplane yaws to the right in still air, it is pushed to the right and started off on a right turn. We saw above that the natural banking is suitable for the turn. In gusty air, if the apparent wind shifts 10° to the left the lateral force pushes the aeroplane to the right.

Numerical values are interesting. Suppose a plus yaw of 10° in still air. The rolling moment at high speed is 2,000 pounds-feet. This is equivalent to a down load of 55 pounds on the right aileron and an up load of 55 pounds on the left aileron. The pilot with his aileron control can, if he wish, produce a rolling moment over three times this magnitude, so that he can prevent the aeroplane taking charge and hold it level. Approaching a landing, it is most important that the aileron control shall be very powerful compared with the natural banking tendency. Excessively stable aeroplanes may be really dangerous to land in gusty air. In any aeroplane design, the relative magnitudes of the natural rolling moment and the aileron control available should be carefully considered.

If the aeroplane side slip with a lateral velocity v, the resultant velocity of the center of gravity of the aeroplane is obtained by combining v as a vector with the forward speed U. The apparent wind in still air is then inclined to the axis of the aeroplane as it would be were the aeroplane yawed from her course by an angle

$$-\psi = \tan^{-1}\frac{v}{U}.$$

A side slip to the left is equivalent aerodynamically to a positive or right-hand yaw. The sign of the lateral force Y is negative for a plus yaw and hence resists the side slip, as is desired.

The asymmetrical motion is a combination of rolling, yawing, and side slipping as is indicated by the qualitative discussion given above and by the equations of motion in Part I, §9. In order that, under the influence of N, L, and Y, acting in concert, the disturbed motion shall be stable, the aeroplane must tend to return in time to its original attitude. It is impossible to determine whether the aeroplane is thus stable from a consideration of N, L, and Y separately. The term "directional stability," frequently used, means very little with regard to the probable motion of the aeroplane.

The quantitative determination of the stability of the motion can be made only after we have found the numerical values of the coefficients needed in the equations of motion in Part I, §9.

§2. RESISTANCE DERIVATIVES

The rates of change of N, L, and Y with velocity of side slip v are the partial derivatives N_v, L_v, Y_v. The side slip velocity v is equivalent to an angle of yaw ψ given by:

$$\tan\psi = -\frac{v}{U}.$$

If ψ is small and measured in degrees, the tangent is equal approximately to the circular measure of the angle, or

$$\psi = -\frac{1}{57.3}\frac{v}{U},$$

and

$$N_v = \frac{\partial N}{\partial v} = -\frac{57.3}{U}\cdot\frac{\Delta N}{\Delta\psi}.$$

The fraction $\frac{\Delta N}{\Delta\psi}$ is the slope of the curve of N plotted on angle of yaw ψ as abscissæ.

Similarly:

$$L_v = -\frac{57.3}{U}\cdot\frac{\Delta L}{\Delta\psi},$$

and

$$Y_v = -\frac{57.3}{U}\cdot\frac{\Delta Y}{\Delta\psi}.$$

Taking the slopes of the curves of L, N, Y at $\psi=0$ from figure 15, we obtain the following "resistance derivatives" needed in the lateral equations of motion.

High speed:

$$i=0^\circ\begin{cases} Y_v = -.204, \\ L_v = +3.06, \\ N_v = -.449. \end{cases}$$

Intermediate speed:

$$i=6^\circ\begin{cases} Y_v = -.0878, \\ L_v = +3.44, \\ N_v = -.351. \end{cases}$$

Slow speed:

$$i=12^\circ\begin{cases} Y_v = -.106, \\ L_v = +1.91, \\ N_v = -.53. \end{cases}$$

Note that these derivatives do not change greatly with speed. In the longitudinal motion the effect of change of speed (attitude) was more marked.

§3. ROLLING MOMENT DUE TO YAWING, L_r

It is obvious that if an aeroplane yaws quickly, the outer wing tip moves through the air more rapidly than the inner wing tip and, hence, due to the spin, the lift on the outer wing is the greater. The resultant rolling moment tends to bank the aeroplane suitably for the turn. The magnitude of this rolling moment was in dispute in the recent Curtiss-Wright patent litigation. The following calculation leads to a simple formula to determine the roll due to angular velocity in yaw.

In our notation, a rolling moment L is expressed in pounds-feet per unit mass. In pounds-feet on the aeroplane, the moment is mL, where m is the mass W/g in slugs.

The derivative L_r is the rate of change of rolling moment with an angular velocity in yaw of r radians per second, or

$$\frac{\partial L}{\partial r}=L_r.$$

Let U = the velocity of advance of the center of gravity of the aeroplane in feet per second. U is a negative number.

S = span of the aeroplane (one plane) in feet.

b = chord of one plane in feet.

$W/g=m$ = mass of aeroplane in slugs.

r = angular velocity of yaw in radians per second, positive for a right-hand turn.

Consider an element of wing area on the left wing of width dy in the y axis and depth b in x axis. The distance from the center of gravity of the aeroplane to the center of this element is y feet, positive for the left wing.

The velocity through the air of this element is $U-yr$, since the increase of air speed due to spin is yr.

If we assume that the lift of the wings is equal to the weight of the aeroplane, we neglect the small vertical forces on body and tail only.

The lift in pounds per square foot per foot-second velocity is the usual "lift coefficient" for the wing, which can be computed from the model tests for Z. Thus:

$$K=\frac{Z_0 m}{AU^2}.$$

Where:

$A=265$, the total area of both wings.

Then the lift in pounds on the elementary strip of wing of area bdy is

$$Kbdy(U-yr)^2.$$

The rolling moment on the aeroplane of this elementary lift force is

$$Kbydy(U^2-2Uyr+y^2r^2),$$

and the total rolling moment on one whole plane is,

$$Kb\int_{-\frac{S}{2}}^{+\frac{S}{2}}(U^2-2Uyr+y^2r^2)ydy.$$

But $\int_{-\frac{S}{2}}^{\frac{S}{2}} by^2dy=I$, the moment of inertia of the area of one plane, and

$$\int_{-\frac{S}{2}}^{\frac{S}{2}} U^2ydy=0=\int_{-\frac{S}{2}}^{\frac{S}{2}} y^3dy.$$

Hence the rolling moment on one plane is $-2UKIr$, and substituting for K its expression above,

$$-2\frac{Z_0 m}{AU}Ir.$$

For two identical wings of rectangular form, we have for our complete aeroplane a total rolling moment in pounds-feet per unit mass:

$$L=-\frac{1}{6}\frac{Z_0 S^2}{U}r, \text{ making } Z_0=g,$$

$$L_r=-\frac{gS^2}{6U} \text{ for horizontal flight.}$$

It appears that L_r can be made small by short span and high speed. The sign of L_r is such that the bank is proper for the turn.

Numerically, we have, making the mean span $S=40.2$ feet and $b=5.62$ feet,

$$\begin{aligned} L_r &= -8660/U, \\ &= +77.0, \text{ high speed, } i=0°, \\ &= +132.5, \text{ intermediate speed, } i=6°. \\ &= +160.0, \text{ slow speed, } i=12°. \end{aligned}$$

Note that L_r (which is unfavorable to "spiral" stability) becomes larger at low speed.

§4. YAWING MOMENT DUE TO ROLLING, N_p

When an aeroplane rolls with an angular velocity p radians per second (positive when right wing goes down), an elementary area of the right wing has its angle of incidence increased and a corresponding element of the left wing has its angle of incidence diminished by the same amount.

If p is small, the resultant air velocity at a point y feet from the center line is

$$\sqrt{U^2+p^2y^2}=U, \text{ neglecting } p^2.$$

On the right wing, the angle of incidence at any point is increased by a small angle α, given by $\tan\alpha=py/U$. Due to the greater angle of incidence, the head resistance of the element is increased.

On a curve of the "drift coefficient" for the wing shape (see fig. 3, Part I) we may draw a tangent line at the point on the curve corresponding to the angle of incidence for normal flight. For small changes in incidence from normal incidence, we may substitute this tangent line for the actual curve without material error. The value of the drift coefficient in pounds per square foot per foot-second is then

$$K_{x_1} = K_{x_0} + \sigma a,$$

where K_{x_0} is the coefficient when i is the normal angle, σ is the slope of the tangent line and a the small change in incidence defined above. The slope σ is conveniently measured in units of K_x change per degree angle. If the subtangent or projection of the tangent line is j degrees,

$$\sigma = \frac{K_{x_0}}{j},$$

and

$$K_x = K_{x_0} + K_{x_0} \frac{a}{j}.$$

The head resistance of an element of the right wing is

$$-bdyK_xU^2 = -\left(K_{x_0} + K_{x_0}\frac{a}{j}\right) bdyU^2,$$

and the yawing moment on the aeroplane due to it is

$$+\left(K_{x_0} + K_{x_0}\frac{a}{j}\right) bU^2ydy.$$

But $\tan a = \frac{py}{U}$ or $a = 57.3\frac{py}{U}$, if a is small. Then the total yawing moment on a single plane is

$$\int_{\frac{S}{2}}^{-\frac{S}{2}} \left(K_{x_0} + K_{x_0}\frac{a}{j}\right) bU^2ydy.$$

The integral of the first term is zero, and the second term reduces to

$$-\frac{57.3UK_{x_0}I}{j}p,$$

where I is moment of inertia of one plane. For a biplane of two rectangular wings, the total yawing moment in pounds-feet is

$$mN = \frac{-57.3UK_{x_0}bS^2}{6j}p.$$

Hence:

$$N_p = \frac{-57.3UK_{x_0}bS^2}{6jm}.$$

To calculate N_p, we have:

i	U	K_{x_0}	j	b	S	m	N_p
0	−112.5	....	00	5.62	40.2	50	0
6	− 65.3	.0000443	6.0	5.62	40.2	50	+33.5
12	− 54.0	.0001047	6.9	5.62	40.2	50	+57.0

Since U is feet per second, K_{x_0} must be in pounds per square foot per foot-second velocity. Values for the drift coefficient were taken from a curve corrected to apply to full-speed full-scale, aspect ratio 7, and biplane of gap 1.1 times chord.

Note that the positive sign of N_p indicates that for a positive roll (to the right) a yaw to the right is assisted. At high speed the aeroplane flies at a small angle of incidence where the drift curve plotted on incidence is about horizontal. N_p is, therefore, zero at this attitude.

§5. DAMPING OF ROLL, L_p

The wide spreading wings very effectively damp the rolling, and the resisting or damping moment in pounds-feet on the aeroplane is mpL_p for an angular velocity p radians per second in roll.

The method of oscillations previously used to determine the damping of the pitching M_q is applied to determine L_p. Figure 17 (pl. 2) shows the oscillating apparatus set up to impress an oscillation in roll about the center of gravity of the model.

Using a similar notation, the equation of motion of the complete apparatus with model is

$$I\frac{d^2\phi}{dt^2}+(\lambda_0+\lambda_w+\lambda_m)\frac{d\phi}{dt}+(K-Cm')\phi+M_0-M_s=0.$$

Where λ_0 represents damping due to friction, λ_w due to wind on apparatus, and λ_m due to wind on model. The moment of inertia of the entire oscillating mass I is found by a simple experiment.

The solution for points of maximum amplitude is of the form

$$\phi=\phi_0 e^{-\frac{\lambda t}{2I}},$$

or

$$\frac{\lambda t}{2I}=\log_e\frac{\phi_0}{\phi}=\log_e 9,$$

since the ratio $\frac{\phi_0}{\phi}$ is arranged to be as 9 to 1 on the scale for the pencil of light.

The numerical work follows:

Oscillation in Roll

I model and apparatus $=.0399$, $\frac{\phi_0}{\phi}=9$

I apparatus $=.0373$

Test on Bare Apparatus

V, wind velocity, miles	30	20	0
t, seconds	78	98	197
λ	.0021	.00168	.00083
λ_w (less zero)	.0013	.00085	0

TEST ON APPARATUS WITH MODEL

INCIDENCE OF WINGS 0°

V	35	30	24	18	6.8	0
t	7.3?	6.0	8.3	12	30	175
λ	.024?	.0292	.0211	.0146	.0058	.001
λ_0	.001	.001	.001	.001	.001	.001
λ_w	.0014	.001	.001	.0007	.0003	0
λ_m	.022?	.027	.019	.013	.0045	0

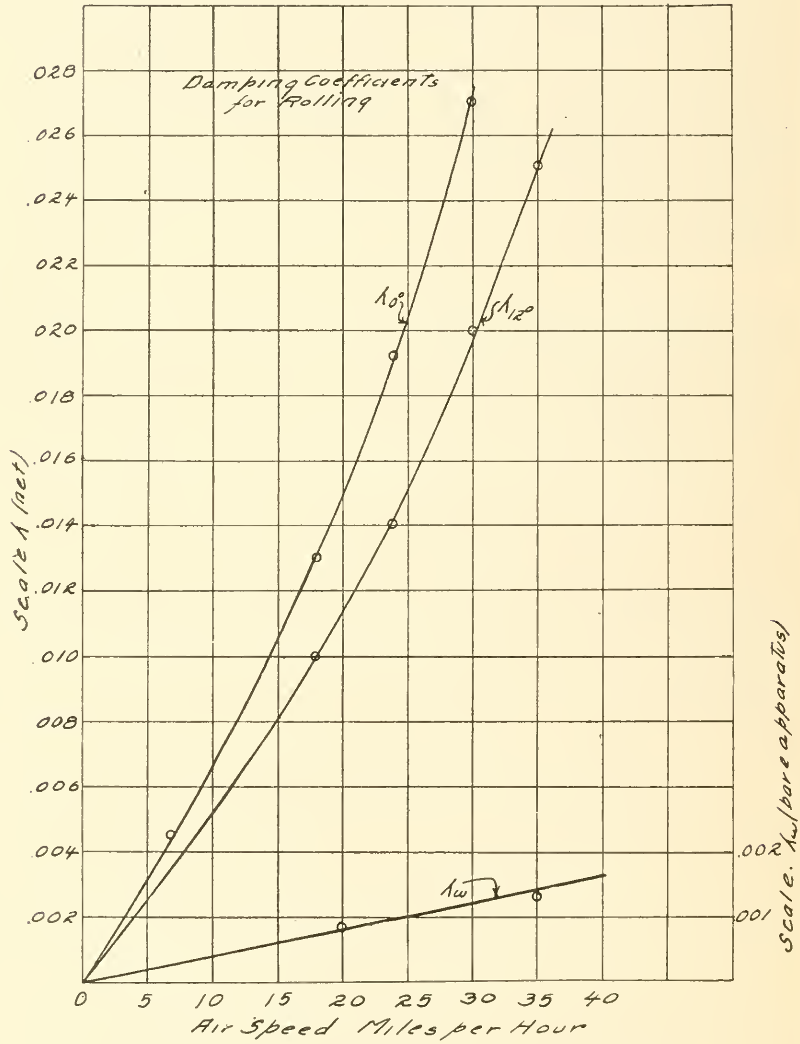


FIG. 16.—Curves of damping coefficient for rolling.

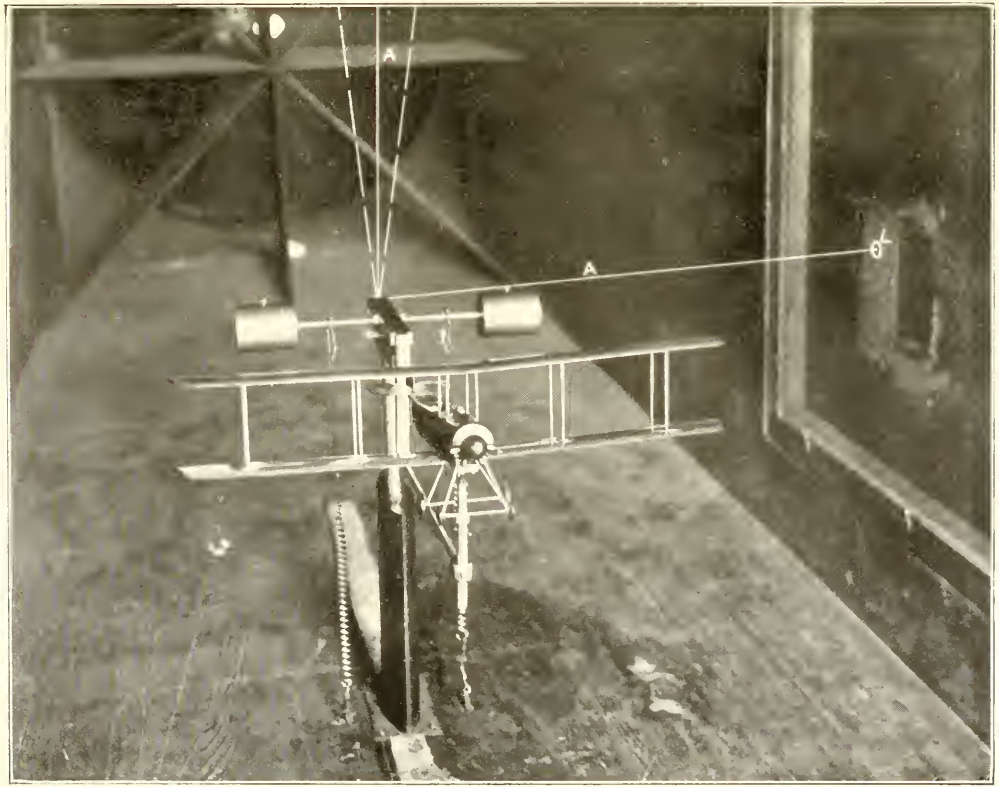


FIG. 17.—MODEL IN POSITION FOR ROLLING OSCILLATION. L, SPECTACLE LENS. AA, PENCIL OF LIGHT

INCIDENCE OF WINGS 12°

V	35	30	24	18	0
t	6.5	8	11	14.5	175
λ	.027	.022	.016	.0121	.001
λ_0	.001	.001	.001	.001	.001
λ_w	.001	.001	.001	.001	0
λ_m	.025	.020	.014	.010	0

The values of λ_w due to wind on apparatus are taken from the curve of λ_w on figure 16 and applied in the calculation to find λ_m net. Figure 16 shows the values of λ_m. It is obvious that the values of λ_m for $i=0°$ at 35 miles per hour is grossly in error. This point is, therefore, rejected.

The curves of λ_m appear to increase more rapidly than the velocity: in fact, a plot on logarithmic paper shows that over the range of wind tunnel speeds λ_m varies approximately as $V^{1.35}$.

Since this damping helps to stop violent rolling, we shall be on the safe side in our stability calculation if we assume that the damping varies directly as the velocity.

To convert λ_m to full scale, we have

$$L_p = \frac{-26^4}{m} \cdot \frac{V}{V_m} \cdot \lambda_m .$$

Where V_m is the speed at which λ_m was measured. Taking the scale factor 26, $m=50$ slugs, $V_m=30$ miles, $V=76.9$ miles for $i=0°$, and $V=36.9$ miles for $i=12°$, we have

$$L_p = -631 = 5.61U, \text{ for high speed,}$$

$$L_p = -224 = 4.15U, \text{ for low speed,}$$

and for the intermediate speed, by interpolation,

$$L_p = -319 = 4.88U.$$

§6. DAMPING OF YAW, N_r

The damping of an oscillation in yaw is probably due to the long body and vertical surfaces at the tail, as well as to the wings. It is not practicable to compute this, and we have employed the same apparatus as before to determine the damping in yaw by the method of oscillations. The model set for the oscillation in yaw is shown on figure 18 (pl. 3).

The equation of motion is similar to that for roll and pitch, thus:

$$I\frac{d^2\psi}{dt^2} + (\nu_0+\nu_w+\nu_m)\frac{d\psi}{dt} + (K-cm')\psi + M_0 - M_s = 0,$$

and

$$\psi = \psi_0 e^{-\frac{\nu t}{2I}}, \text{ or } \frac{\nu t}{2I} = \log_e \frac{\psi_0}{\psi} = \log_e 9.$$

Oscillation in Yaw

I model and apparatus = .0396

I apparatus = .0343

Test on Bare Apparatus

V	35	20	0
t	108	115	120
ν	.0014	.00131	.00126
ν_0	.0013	.00126	.00126
ν_w	.0001	.00005	0

Test on Apparatus with Model

INCIDENCE OF WINGS 0°

V	35	30	24	12
t	52	57	64	105?
ν	.00335	.00306	.00272	.00166?
ν_0	.00126	.00126	.00126	.00126
ν_w	.00013	.00011	.00009	.00004
ν_m	.00196	.00169	.00137	.00036?

INCIDENCE OF WINGS 12°

V	35	30	18	8
t	33	36	47	73
ν	.00528	.00484	.00371	.00239
ν_0	.00126	.00126	.00126	.00126
ν_w	.00013	.00011	.00006	.00003
ν_m	.00389	.00347	.00239	.00110

INCIDENCE OF WINGS 6°

V	35	30	20
t	46	53	71
ν	.00379	.00329	.00245
ν_0	.00126	.00126	.00126
ν_w	.00013	.00011	.00007
ν_m	.00240	.00192	.00112

$$N_r = \frac{26^4}{50} \cdot \frac{V}{V_m} \cdot \nu_w.$$

$N_r = .35U = -39.4$, for $i = 0°$,

$N_r = .398U = -26.0$, for $i = 6°$,

$N_r = .72U = -38.9$, for $i = 12°$.

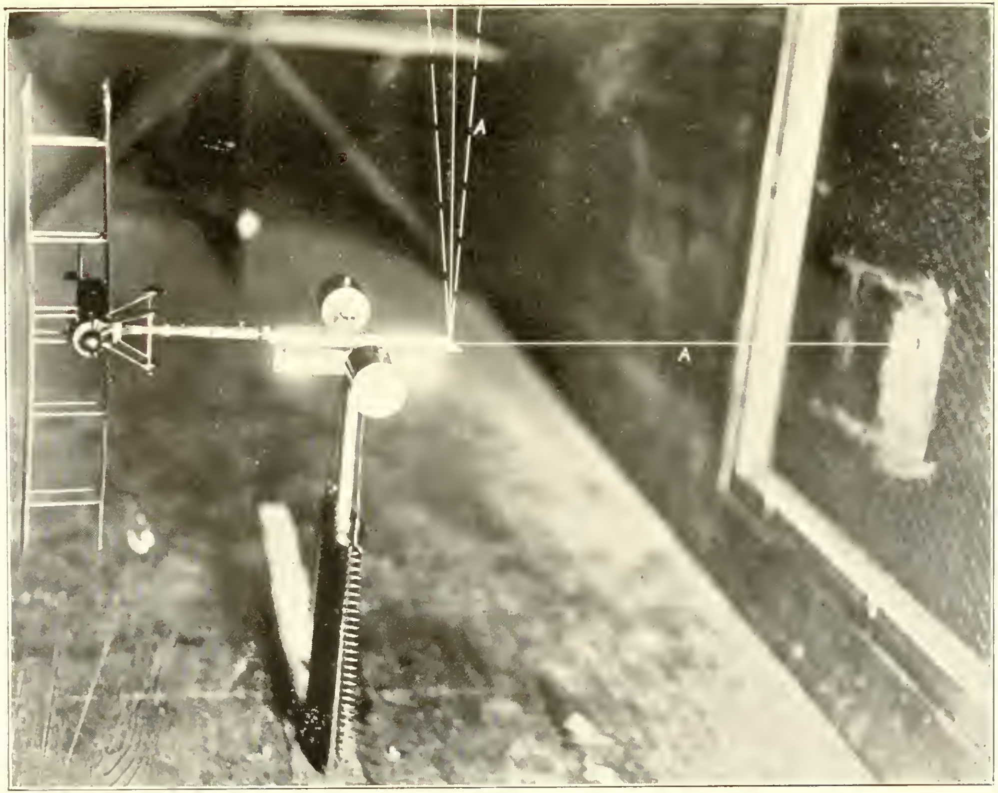


FIG. 18.—MODEL IN POSITION FOR YAWING OSCILLATION. L, SPECTACLE LENS. AA, PENCIL OF LIGHT

The curves of v_m of figure 19 show that the damping of the yaw increases with speed approximately as the first power. The damping of yaw N_r is in magnitude only about $\frac{1}{10}$ the damping of roll L_p. Consequently, the precise determination of N_r is attended with some experimental difficulty.

It is to be noted that N_r diminishes with the velocity, while at the same time it increases with the angle of attack. The value of N_r at

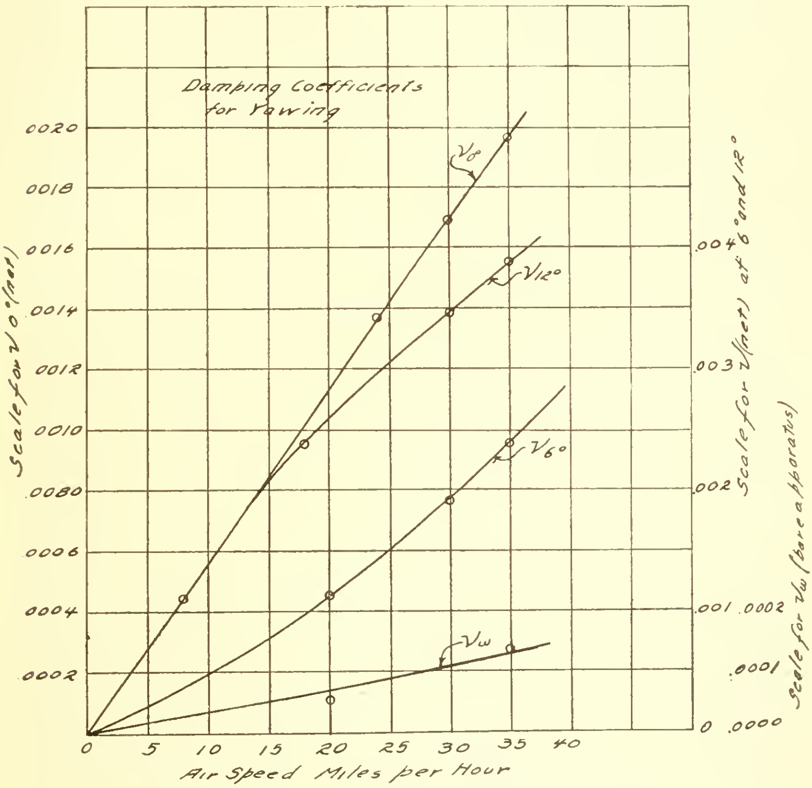


FIG. 19.—Curves of damping coefficient for yawing.

high speed .35U is practically equal to its value at low speed .72U. It seems reasonable to expect that at large angles of incidence the damping of yaw due to the wings would be much greater than at small angles were the speed the same.

For the intermediate speed $i=6°$ the coefficient N_r is least. This is due to the fact that from 0° to 6°, U drops from -117.5 to -65.3 feet per second, while from 6° to 12° U drops very little more: only from -65.3 to -54 feet per second.

§7. NEGLECTED COEFFICIENTS

The changes in lateral force Y due to angular velocity of roll and yaw, represented by the coefficients Y_p and Y_r, are neglected as unimportant. The surface of the aeroplane is fairly symmetrical about the center of gravity and it is unlikely that any appreciable lateral force could be created by any small angular velocity p or r. In the calculations to follow Y_p and Y_r are made zero.

The products of inertia are also neglected as not important and difficult to estimate for an actual machine.

§8. INDEPENDENCE OF THE LONGITUDINAL AND LATERAL MOTION

It is seen on figure 20 that the values of X, Z, and M are somewhat changed as the aeroplane yaws, and to this extent it is not strictly correct to consider the lateral motion separately. We may imagine that if there be set up a combined oscillation about the flight path in roll, yaw, and side slip, the aeroplane will be influenced to take up an oscillation in pitch of the nature of a forced oscillation. However, any oscillation in pitch has already been shown to die out rapidly (since the longitudinal motion is stable and strongly damped). We may then consider the pitching induced by yawing, etc., as of the same nature as that caused by any accidental disturbance of longitudinal equilibrium, such as might result from gusty winds, shifting of weights, or the firing of a gun. If the longitudinal motion be stable, that stability should be quite independent of the nature of any disturbing agent which gives the initial amplitude to the oscillation, provided the phenomenon of resonance is not present. That is, if the natural period of the lateral motion, if oscillatory, happen by some remote chance to be equal to the natural period of the longitudinal oscillation, it may be possible for a machine which is unstable laterally to seriously compromise its longitudinal stability.

If the lateral motion be stable and, if oscillatory, damp out quickly, it is difficult to see how any marked disturbance of the longitudinal motion can be induced by the lateral motion.

In circling flight, there is a constant angular velocity of yaw and probably some side slip. In this case, the lateral and longitudinal motions are interdependent, and the methods of calculation of this paper will not apply. Indeed, we should have to combine the six general equations of motion giving rise to a single equation of the eighth order, which must then be solved for all the roots. In the present state of our knowledge, the calculation of the stability of circling flight appears impracticable.

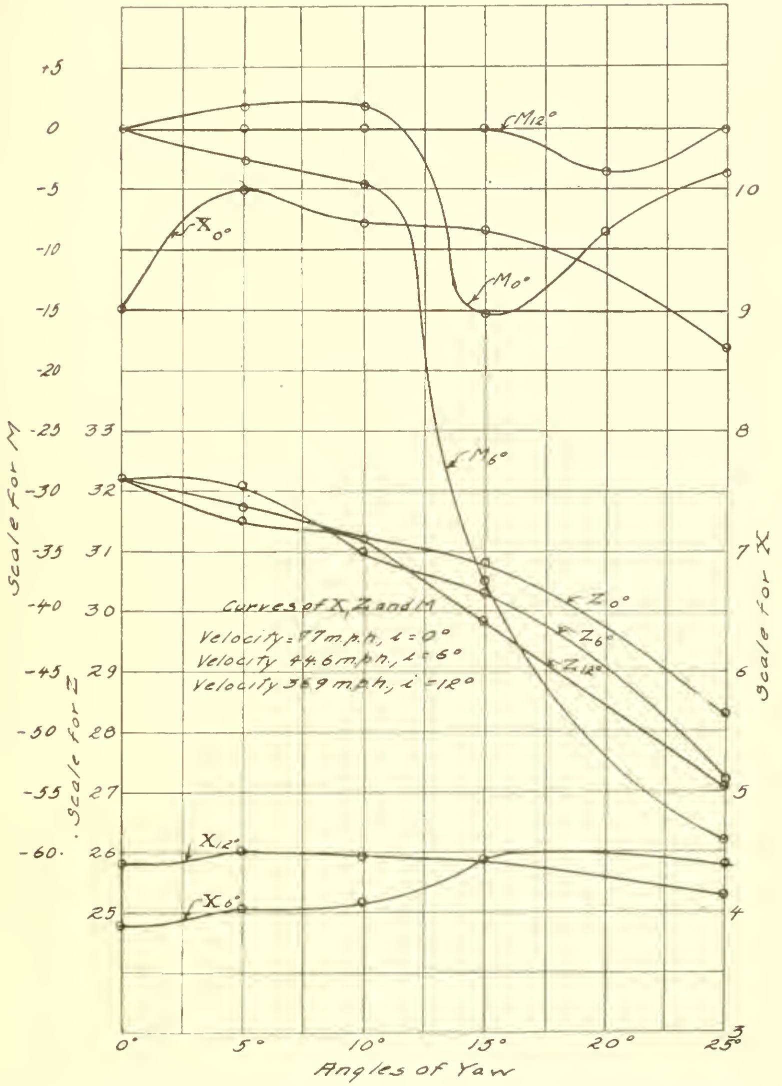


FIG. 20.—Curves of normal force, longitudinal force, and pitching moment as angle of yaw changes.

For flight in a straight line, we may reasonably conclude that if the lateral motion be stable it will not compromise the stability of the longitudinal motion, and *vice versa.* Such a machine should, in still air, follow its trajectory without the aid of the pilot. In gusty air, it would roll and pitch and yaw as well as side slip and rise and sink, but, if the altitude be great, there should be no danger. The machine would not follow a fixed course, if controls were abandoned, but would adjust its trajectory constantly to the changing conditions of the air in an effort to maintain the same relative velocity through the air and the same angle of incidence.

On the other hand, if the lateral motion be unstable and the angle of yaw become as great as 10°, the curves of figure 16 show that the head resistance X is not greatly changed for slow-speed attitudes and increases but 10 per cent at high speed. This should tend to slow down the aeroplane very little.

The change in Z, or lift, is insignificant.

However, the change in M is most interesting. For $i=12°$ no change in M is produced by yaw, but for $i=6°$ a small diving moment is induced. For an angle of yaw of 15° or more, this diving moment is enormously increased. For $i=6°$, $\psi=15°$, $mM=37\times 50=1{,}850$ pounds-feet, corresponding to a force on the elevator of nearly 100 pounds.

If the pilot attempt to turn without banking he may side slip so rapidly that he has the relative wind making an angle of 15° to the longitudinal axis of the aeroplane. The aeroplane will then tend to dive sharply. Similarly, an excessive bank may induce a side slip inwards and the same tendency to nose dive. The cause of this tendency to nose dive shown here is not understood, but it is significant that many accidents have occurred to inexperienced pilots in turning.

§9. LATERAL STABILITY, DYNAMICAL

The combined asymmetrical motion in roll, yaw, and side slip will be called " lateral." For simplicity we will consider horizontal flight in a straight line in still air, and for this condition investigate the character of the disturbed motion.

From the detail plans, the radii of gyration K_A and K_C have been calculated. It is assumed that these values are not appreciably changed by change of axes corresponding to the changed attitudes proper for different speeds. K_A and K_C as given are referred to the axes used at high speed. The products of inertia are neglected as unimportant.

From Part I, §9, we obtain the following simplified formulæ for the coefficients of the biquadratic equation which is characteristic of the lateral motion. The quantities Y_r, Y_p, θ are made equal to zero. Then:

$$A_2D^4+B_2D^3+C_2D^2+D_2D+E_2=0.$$

Where:

$$A_2=K_A^2K_C^2,$$
$$B_2=-Y_vK_A^2K_C^2-K_C^2L_p-K_A^2N_r,$$
$$C_2=(N_rL_p-L_rN_p)+Y_vL_pK_C^2+K_A^2(N_vU+N_rY_v),$$
$$D_2=-Y_v(N_rL_p-L_rN_p)+U(N_pL_v-N_vL_p)+gK_C^2L_v,$$
$$E_2=g(N_vL_r-L_vN_r).$$

These coefficients may now be calculated from the known constants of the aeroplane, and Routh's discriminant, $B_2C_2D_2-A_2D_2^2-B_2^2E_2$, found. The condition that the motion shall be stable is that A_2, B_2, C_2, D_2, E_2 shall each be positive as well as Routh's discriminant.

The numerical work is laborious and the results only are given in the table.

Coefficients Affecting Lateral Motion

	High speed	Intermediate speed	Low speed
Angle of incidence, i	0°	6°	12°
Velocity, ft.-sec., U	−112.5	− 65.3	− 54.0
Mass, slugs, m	50.0	50.0	50.0
K_A	5.2	5.2	5.2
K_C	6.975	6.975	6.975
Y_v	− .204	− .0878	− .106
L_v	+ 3.06	+ 3.44	+ 1.91
N_v	− .449	− .351	− .53
Y_p	0	0	0
L_p	−631.0	−319.0	−224.0
N_p	0	+ 33.5	+ 57.0
Y_r	0	0	0
L_r	+ 77.0	+132.5	+160.0
N_r	− 39.4	− 26.0	− 38.9
A_2	1310.0	1310.0	1310.0
B_2	31830.0	16350.0	12090.0
C_2	3274.0	5910.0	1630.0
D_2	41780.0	5490.0	3490.0
E_2	2770.0	1386.0	−335.0
$B_2C_2D_2-A_2D_2^2-B_2^2E_2$	37400×10^9	123×10^9	3.7×10^9
Character of motion	Stable	Stable	Unstable

It is seen that, for the particular aeroplane under consideration, Routh's discriminant and the coefficients of the biquadratic are all positive at high and intermediate speeds. The motion in these two cases is, therefore, stable.

At low speed, however, we observe that E_2 becomes negative, indicating that the lateral motion is unstable. That is to say, one at least of the roots of the biquadratic increases with time. In this case Routh's discriminant continues to be positive, but is small compared with its value at high speed.

It is unfortunate that this lateral instability is associated with the longitudinal instability which was found in Part I to be present at low speed.

§10. CHARACTER OF LATERAL MOTION

Bairstow has shown that for the usual values of the coefficients of the biquadratic equation for the lateral motion, the equation in question may be factored approximately, giving:

$$\left(D+\frac{E_2}{D_2}\right)\left(D+\frac{B_2{}^2-A_2C_2}{A_2B_2}\right)\left(D^2+\left(\frac{C_2}{B_2}-\frac{E_2}{D_2}\right)D+\frac{B_2D_2}{B_2{}^2-A_2C_2}\right)=0,$$

provided E_2 is small compared with B_2 or D_2, and $B_2D_2-C_2$ is small compared with $C_2{}^2$.

In our cases, the second condition is not satisfied but the error made is found by trial solutions to be unimportant.

High Speed.

Thus for the high-speed condition:

$$\text{First factor, } D=-\frac{E_2}{D_2}=-.0665.$$

This is a subsidence which tends to reduce the amplitude of an initial disturbance to half value in $t=\frac{0.69}{.0665}=10.4$ seconds. We may consider this motion fairly stable.

For the second factor we have another subsidence given by

$$D=-\frac{B_2{}^2-A_2C_2}{A_2B_2}=-23.2,$$

which reduces to half value in $t=\frac{0.69}{23.2}=.03$ second. Such motion is so heavily damped that it would never be observed on the aeroplane.

The third factor gives upon substitution:

$$D^2+\left(\frac{C_2}{B_2}-\frac{E_2}{D_2}\right)D+\frac{B_2D_2}{B_2{}^2-A_2C_2}=D^2+.967D+1.375=0,$$

or

$$D=-.484\pm 1.07i.$$

This is a pair of imaginary roots indicating an oscillation of natural period $p = \frac{2\pi}{1.07} = 5.9$ seconds, which is damped to half the initial amplitude in $t = \frac{0.69}{.484} = 1.4$ second. The motion is so heavily damped as to be of no consequence. The period is fairly rapid, and if the damping were not great, the oscillation might become uncomfortable.

For the high-speed case, it appears that the lateral motion is quite stable.

Intermediate Speed.

At the intermediate speed, where $i = 6°$, we have for the first factor:

$$D = -.252,$$

a subsidence which damps to half amplitude in

$$t = \frac{0.69}{.254} = 2.72 \text{ seconds.}$$

This motion is very strongly damped, even more than at the high speed.

Similarly, the second factor gives an enormously damped subsidence.

$$D = -12.1,$$

$$t = \frac{0.69}{12.1} = .057 \text{ second.}$$

The oscillation corresponding to the third factor is of fairly slow period, but so strongly damped that it is of slight importance. Thus:

$$D^2 + .11D + .346 = 0,$$

$$D = -.55 \pm .586i,$$

$$p = \frac{2\pi}{.586} = 10.7 \text{ seconds' period,}$$

$$t = \frac{0.69}{.55} = 1.25 \text{ second to damp 50 per cent.}$$

Slow Speed.

For the slow-speed condition, $i = 12°$, we observed that the coefficient E_2 is negative indicating instability of motion. Mathematically, that is to say, the real root corresponding to the first factor of Bairstow's approximate method,

$$D = -\frac{E_2}{D_2} = .096,$$

is now no longer a subsidence, but a divergence which doubles itself in $t = \frac{0.69}{.096} = 7.2$ seconds. This is not an alarming rate of increase, since 7 seconds should be ample time for a pilot to observe a deviation from normal attitude and to correct it by use of his controls. However, the aeroplane could only be flown at this speed even in still air provided the pilot were alert.

The second factor is a strongly damped subsidence $D=-9.12$, which damps to half amplitude in .08 second.

The third factor is an oscillation,

$$D^2+.231D+.292=0,$$
$$D=-.116\pm.528i,$$

having a period of $\frac{2\pi}{.528}=12$ seconds, which is damped to half amplitude in $t=\frac{0.69}{.116}=6$ seconds. This oscillation is stable, but the damping is only moderate, and it may well be felt on the aeroplane in flight. In some types of aeroplane, it is likely that this motion may be undamped and hence the amplitude of successive oscillations will be increasing, giving rise to instability of a new character.

§11. THE "SPIRAL DIVE"

The motion found corresponding to E_2 negative, as at slow speed, may be traced to the resistance derivatives involved in the expression for E_2. Thus:

$$E_2=g(N_vL_r-L_vN_r),$$

and E_2 will be positive only when L_v/N_v is greater than L_r/N_r. For stability, or E_2 positive, L_v and N_r should be large and N_v and L_r small.

The derivative L_v depends on the rolling moment due to side slip and can be made large and positive by an upward dihedral angle to the wings or by vertical fin surface above the center of gravity of the aeroplane. At low speed and high angle of incidence we see that L_v is diminished. Thus, at 6° and 44.6 miles, $L_v=3.44$, while at 12° and 36.9 miles, $L_v=1.91$. The drop in speed is only about 18 per cent. Hence the drop in L_v cannot be due to the lower speed, but must be due to the greater angle of incidence.

Let i be the angle between the wind direction and the center line of the wings where yaw ψ is zero. Let ϕ be the angle through which each wing tip is raised, and let the angle between the wind direction for a yaw ψ and the plane of the chord of the up wind wing be i'. Then it can easily be shown by geometry that approximately

$$i'=i_0\pm\beta\psi,$$

when i, ψ, and β are small[1] and expressed in circular measure.

[1] A. Fage, "The Aeroplane," p. 82, Griffin, London, 1915.

In our case $\beta = 1\overset{\circ}{.}6$, then for $i = 12^\circ$ and $\psi = 10^\circ$, $i' = 12\overset{\circ}{.}3$, while for $i = 6^\circ$, $i' = 6\overset{\circ}{.}3$. This is an increase of incidence with 10° yaw of but 2.5 per cent at low speed, and 5 per cent at intermediate speed.

Since a side slip is equivalent to a yaw, and since the rolling moment due to side slip is largely caused by greater lift on the wing which is toward the wind, it appears reasonable to conclude that this greater lift is a consequence of the greater angle of incidence. But we see above, by a rough calculation, that the relative increase in incidence on a dihedral wing for given angle of yaw is much greater for the 6° attitude than for the 12° attitude. The falling off of L_v observed experimentally is, therefore, to be expected for an aeroplane with raised wing tips.

A discussion might be opened here as to whether it would not be preferable to use vertical fin surfaces above the center of gravity or a swept back wing (" retreat ") to obtain the desired righting moment L_v on side slip, rather than the dihedral arrangement. Until further experiments have been made, it is not profitable to speculate on this question, but one would see no reason *a priori* to expect the coefficient L_v, given by vertical fins, to depend in any way upon the angle of incidence of the normal flight attitude.

To preserve stability, we must make N_r large also. This coefficient is a measure of the damping of angular velocity in yaw, and can be made great by vertical surface forward and aft of the center of gravity. A rectangular body with flat sides, vertical fin surface at the tail (rudder), and the increased drift on the forward moving wing all combine to resist or damp the spin in yaw. The designer can, at his pleasure, increase both L_v and N_r by proper fin disposition. Note that N_r is not different at different speeds.

On the other hand, it is necessary to make N_v or the yaw due to side slip small. A preponderance of fin surface aft will make N_v large and is, therefore, dangerous. A machine that shows strong " weather helm " or has great so-called directional stability is likely to be unstable because the large N_v may make E_2 negative. The vertical fin surface should be fairly well balanced fore and aft, and directional restoring moments should not be great. Note that N_v does not vary much with different speeds.

The derivative L_r is characteristic of the rolling moment due to velocity of yaw or spin and was shown to be caused by the greater air speed on the outer wing in turning. It is not generally possible for a designer to make L_r small, though a short span will help matters.

Note that L_r is greatest at low speed and high angle of incidence. It should be unaffected by dihedral angle of wings.

The instability corresponding to E_2 negative is, therefore, a tendency on side slip to the right, for example, to head to the right toward the relative wind on account of much fin surface aft. At the same time, due to the spin in yaw, the machine tends to overbank on account of the greater lift on the left wing. The increased bank, increases the side slip, the yaw becomes more rapid and in turn the overbanking tendency is magnified. The aeroplane starts off on a spiral dive and will spin with greater and greater angular velocity. The term " spiral instability " has been given to this motion.

Spiral instability appears to be the most probable form of instability present in an ordinary aeroplane. It appears to be readily corrected by modification of fin surface and there appears to be no excuse for leaving it uncorrected. It is true that an alert pilot should have no trouble in keeping an aeroplane out of a spiral dive, but in case of breaking of a control wire disaster would be certain if the machine were spirally unstable.

§12. "ROLLING"

The second approximate factor

$$D + \frac{B_2^2 - A_2C_2}{A_2B_2} = 0,$$

when A_2C_2 is small compared with B_2^2, is seen to reduce to:

$$D + \frac{B_2}{A_2} = 0.$$

or

$$D = -\frac{B_2}{A_2} = +Y_v + \frac{L_p}{K_A^2} + \frac{N_r}{K_B^2}.$$

Now Y_v, L_p, and N_r may be expected to be always negative in ordinary machines, and the radii of gyration K_A and K_B are essentially positive. Hence this root D will always be negative and the motion a damped subsidence. It will be observed that Y_v expresses resistance to side slip, L_p damping of an angular velocity in roll due to the wings, and N_r damping of an angular spin in yaw. In magnitude L_p is usually so great that Y_v and N_r may be neglected, giving roughly

$$D = \frac{L_p}{K_A^2} = -\frac{224}{27} = -8.3$$

at low speed, or a subsidence damped 50 per cent in $t = .08$ second.

The more exact calculation made in §11 showed $t=.076$ second. In a machine of very short span and great moment of inertia in roll, we might expect $\frac{L_p}{K_A^2}$ to become small, but never positive so long as forward speed is maintained.

When an aeroplane is at such an attitude that further increase in angle of incidence produces no more lift ("stalled"), the damping of a roll by the wings L_p may vanish. Then the downward moving wing, although its angle of incidence be increased, has no additional lift over the other and, hence, there is no resistance to rolling. In this critical attitude, pilots have reported that the lateral control by ailerons has no effect and the aeroplane is unmanageable.

In any reasonable attitude short of stalling, there appears to be no reason to fear instability in "rolling" corresponding to this second factor of the equation.

§13. THE "DUTCH ROLL"

In the approximate solution of the biquadratic, the third factor,

$$D^2+\left(\frac{C_2}{B_2}-\frac{E_2}{D_2}\right)D+\frac{B_2D_2}{B_2^2-A_2C_2}=0,$$

for most machines will have A_2C_2 small compared with B_2^2, and we may write:

$$D^2+\left(\frac{C_2}{B_2}-\frac{E_2}{D_2}\right)D+\frac{D_2}{B_2}=0.$$

Considering the usual magnitudes of the derivatives entering in B_2, C_2, D_2, E_2, we may write very approximately:

$$B_2=-K_C^2L_p,$$
$$C_2=(N_rL_p-L_rN_p),$$
$$D_2=+gK_C^2L_r,$$
$$E_2=g(N_vL_r-L_vN_r).$$

The motion is damped and stable, provided $\frac{C_2}{B_2}-\frac{E_2}{D_2}$ is positive, and the period

$$p=\frac{2\pi}{\frac{1}{2}\sqrt{\frac{4D_2}{B_2}-\left(\frac{C_2}{B_2}-\frac{E_2}{D_2}\right)^2}}.$$

or approximately $=p=2\pi\sqrt{\frac{B_2}{D_2}}$.

Since $\sqrt{\frac{B_2}{D_2}}$ is ordinarily of the order of 1 or 2 the period may be of the order of 6 or 12 seconds. This period is rapid compared with that of the longitudinal motion and unless strongly damped, the motion may become so violent as to be uncomfortable. Note that since N_v, L_v, N_p, L_p, N_r, L_r are involved, the motion must consist of a combination of side slipping, rolling, and yawing.

The motion is stable and the oscillation tends to damp out in time and the aeroplane to return to her course if $\frac{C_2}{B_2} - \frac{E_2}{D_2}$ is positive. To damp to half amplitude requires $t = \dfrac{0.69}{\frac{1}{2}\left(\frac{C_2}{B_2} - \frac{E_2}{D_2}\right)}$ seconds.

Substituting approximate expressions we have

$$\frac{C_2}{B_2} - \frac{E_2}{D_2} = \frac{L_r}{K_C^2}\left(\frac{N_p}{L_p} - \frac{N_v}{L_v}\right).$$

Since L_r is positive, in order for the damping to be real, $-N_v/L_v$ must be greater than N_p/L_p and positive.

Stability of this motion is, therefore, assisted by:

1. Large negative yawing moment due to side slip ("weather cock" stability) N_v. This is incompatible with stability against a "spiral dive."

2. Large damping of the rolling due to rolling L_p.

3. Small positive rolling moment due to side slip L_v. This is also incompatible with stability against the "spiral dive."

4. Small yawing moment due to rolling N_p.

5. Large rolling moment due to yawing velocity L_r; another requirement incompatible with "spiral" stability.

6. Small radius of gyration K_c in yaw.

It does not appear practicable to make N_p small on account of the steepness of the drift curve at high angles of incidence. The drift of the downward moving wing when the aeroplane rolls is increased while the drift of the rising wing is decreased. The resultant yawing moment tends to swing the aeroplane away from her course. Note that at slow speed, near stalling angles, N_p becomes large. This is not desirable, but is unavoidable.

The rolling is heavily damped by the wings and L_p will always be large and negative. This assists stability.

To avoid "spiral" instability, we saw above that it was necessary to make the weather cock or "directional stability" small. That is, N_v was to be small and the preponderance of vertical fin surface aft slight. In the motion now under discussion, we wish to make N_v large. The two conditions imposed are unfortunately conflicting. We must compromise and make N_v numerically not too great, but still essentially negative.

In a similar manner, the rolling moment, due to side slip, or restoring moment, such as is given by high fins or raised wing tips, should be large to avoid "spiral" instability. In the present case, however, we wish to make L_v small.

Likewise the natural banking due to spin in yaw we wish small for "spiral" stability, but we now wish to have this coefficient large.

The conflicting nature of the requirements for stability is here shown by the use of rather drastic simplifications in the more exact formulæ. For the analysis of stability the exact formulæ are easily applied, and the present approximate forms are deduced only in order to trace the effect on the motion of such changes as the designer may be tempted to make on a machine.

It is believed that an excessive dihedral angle upwards is not a cure-all for stability problems. Indeed, in practice, aeroplanes with a large dihedral angle for the wings have been found so violent in their motion under certain circumstances that the average pilot has a firm prejudice against the use of such a wing arrangement. That this prejudice has some physical basis has been shown here. A dihedral angle machine is not likely to run into a "spiral dive," but it is very likely to be unstable on what we may term a "Dutch roll," from analogy to a well-known figure of fancy skating.

We may imagine an aeroplane to yaw to the right accidentally. Due to L_r and L_v the aeroplane banks in a manner proper for the turn, but the roll is retarded by the large damping due to L_p. The turn is assisted by the increased drift on the lower wing due to N_p, and were it not for the much discussed "weather helm" given by N_v, the aeroplane would run off on a right turn. However, N_v tends to turn the aeroplane back to her course. If N_v be sufficient, the machine will swing back to her course and the bank will flatten out. But since the moment of inertia in yaw is considerable, the machine will swing past her course and start on a turn to the left. This swinging to right and left of her course is accompanied by rolling outward and some side slipping.

The analogy to a "Dutch roll" on skates is obvious. If the skater lean too far out he may fall, and if the aeroplane roll too far on the side swings it may happen that the motion will become unstable. If the air be gusty it is very likely that such an aeroplane may be caught on the roll by a side gust and capsized.

The "Dutch roll" in ordinary aeroplanes (which are "spirally" unstable) is not likely to be present, since there is no dihedral and a large rudder. The average pilot would much prefer to deal with a machine which tended to swing down into a "spiral dive" if left to itself because there is no oscillation of rapid period involved.

The production of a laterally stable aeroplane is attendant with many compromises, and it cannot be too strongly insisted upon that a freak type designed to be "very stable" is likely to be rapid and violent in its motion, and even if stable against a "spiral dive" to be frankly unstable against the "Dutch roll."

One may inquire whether a machine made directionally neutral can be made stable. In the notation here used N_v would be approximately zero. The condition that "spiral" instability be not present is:

$$L_v/N_v > L_r/N_r.$$

But for N_r zero, we need only make L_v slightly positive to insure stability in this motion. L_v may be made positive by a very slight preponderance of fin surface above the center of gravity, raised wing tips, etc.

However, in the approximate criterion for stability in the "Dutch roll," we have

$$-N_v/L_v > N_p/L_p,$$

and for N_v zero, the motion is clearly unstable unless the magnitude of the neglected terms is greater than N_p/L_p, which is unlikely.

Replacing neglected terms in C_2, we obtain as a more nearly exact expression:

$$\left(\frac{C_2}{B_2} - \frac{E_2}{D_2}\right) = \frac{L_r}{K_C^2}\left(\frac{N_p}{L_p} - \frac{N_v}{L_v}\right) - Y_v - \frac{K_A^2}{K_C^2}\frac{N_v}{L_p}U.$$

If we make N_v very small as in the case under analysis, the last term vanishes as well as the second, and we have as a condition for $\frac{C_2}{B_2} - \frac{E_2}{D_2}$ positive:

$$-Y_v > \frac{L_r}{K_C^2}\left(\frac{L_p}{N_p}\right).$$

Substituting numerical values for the derivatives, for the slow-speed condition, we find

$$-Y_v = +.106,$$

and

$$\frac{L_r}{K_C^2}\frac{N_p}{L_p} = -\frac{160\times 57}{48.6\times 224} = -.856.$$

The slow-speed motion would, therefore, be very unstable if N_v were zero. Consideration of the magnitude of the derivatives leads us to the conclusion that in any aeroplane, if N_v be made very small, the motion called "Dutch roll" will probably be unstable at low speeds where N_p becomes great.

For high speed, if both N_v and N_p are zero, the lateral motion should be stable regardless of the magnitude of the other derivatives.

With the yawing moment due to rolling as measured by N_p increasing from zero at high speed to $+57$ at low speed, it would seem that, at the maximum speed, any reasonable aeroplane will be stable so far as the "Dutch roll" is concerned, but at low speed it may become unstable in this particular motion.

In general, for high speed, considering the two possible kinds of lateral instability, it is believed that very slight modifications in fin disposition will suffice to render any ordinary aeroplane laterally stable. Likewise, at high speed, longitudinal stability is easily secured. At low speed, the longitudinal motion tends to become unstable as well as one or the other kind of lateral motion.

§14. COMPARISON WITH OTHER AEROPLANES

Any stability discussion is much more suggestive if several aeroplanes can be analyzed in parallel. The only published information on lateral stability is Bairstow's investigation of the Blériot monoplane used above in connection with the longitudinal stability discussion. This monoplane had only a very small rolling moment due to side slip $L_v = .83$ as against $L_v = 3.06$ for the Clark aeroplane for high speed. The coefficient N_v, yawing moment due to side slip, is not greatly different in the two machines. The other coefficients are of the same order of magnitude, except L_p, the damping of a roll, which is small in the monoplane on account of the small wings of short span.

Without further knowledge, we should expect the Blériot to be stable on the "Dutch roll" on account of the small L_v. Bairstow finds a period of 6.5 seconds damped to half amplitude in 1.65 second.

On the other hand, the small L_v would lead us to suspect the sta bility of the spiral motion, especially as L_p is also small. In fact, the coefficient E_2 was found to be slightly negative and the aeroplane, in consequence, spirally slightly unstable. The motion is a slow divergence which doubles itself in 68 seconds. This is an extremely slow change and should give no trouble to a pilot. Indeed, the well-known steadiness in flight of this famous aeroplane is in full agreement with the theoretical conclusions. The Blériot makes no claim to lateral stability, but is essentially a steady aeroplane easily controlled. In the "Dutch roll" the Blériot is very strongly damped and hence very stable. The spiral motion is not damped, but is so slow that the stability may be called neutral. The aim of the French school has always been a machine whose lateral stability is neutral so that it will not be thrown about by the wind.

The Curtiss type military tractor tested by us in a manner identical with that described in this paper, was found at high speed to have resistance derivatives of the same order of magnitude as the Clark tractor, except that a large rudder and deep rectangular body make N_r twice as large for the Curtiss, and there being no high fin surface L_r for the Curtiss is small. As would be expected the spiral motion is slightly unstable, tending to double itself in 28 seconds. The "Dutch roll" is very stable, having a period of 5.25 seconds and damping to half amplitude in 1.77 seconds. The machine in flight at high speed should then have the characteristics of the Blériot and be steady and easily controlled. This is, in fact, the general reputation of this type of aeroplane.

At low speed, matters are not so favorable. We have no data for the Blériot at slow speed, but the Clark model is seen to become spirally unstable to such an extent that an accidental deviation doubles itself in 7.2 seconds.

The "Dutch roll" for the Clark model remains stable at low speed, but is somewhat less strongly damped than at high speed. The period is 12 seconds damped to half amplitude in 6 seconds. This motion should be not uncomfortable.

The Curtiss, at low speed, due to falling off of N_v and marked increase in L_r, becomes spirally stable. The spiral motion is a subsidence damped 50 per cent in 3.3 seconds. The wings had no dihedral angle. A separate test[1] made on a single wing without body or tails showed a small rolling moment for an oblique wind indicating a small and positive L_r. At large angles of incidence this effect was considerably magnified. The decrease in N_v (or in the weather helm) at large angles of incidence cannot be laid to the straight wings. Tests on a wing alone show a small negative N_r which is not changed at large angles of incidence.

The increase in L_v and decrease in N_v for the Curtiss aeroplane, favorable to stability of the spiral motion, are unfavorable to stability in the "Dutch roll." Furthermore, N_p increases from zero at high speed to $+38$ at the low speed, and L_p decreases from -314 to -78. These changes are very unfavorable and, as we should expect, the "Dutch roll" for the Curtiss is unstable. The natural period is about 5.7 seconds and any initial amplitude is doubled in 7.66 seconds.

[1] Smithsonian Misc. Coll., Vol. 62, No. 4. "Experiments on a Dihedral Angle Wing," J. C. Hunsaker and D. W. Douglas.

The motion is a swaying of the aeroplane of increasing amplitude and intensity. However, we must always point out that an alert pilot with powerful controls can check the natural motion of the aeroplane before it has became violent and so maintain his equilibrium.

The increase in N_p at low speed or rather large angle of incidence is due to the steeper drift curve for a wing at large angles. As the aeroplane rolls, the downward moving wing has its drift relatively more increased as the normal flight attitude requires a larger angle of incidence.

The drop in L_p is due to the less steep lift curve at high angles of incidence. As the aeroplane rolls, the increase in angle of incidence of the downward moving wing gives very little increase in lift on that wing if the wing be already near its angle of maximum lift. We might imagine an aeroplane flying at an angle of incidence giving the maximum lift. Any increase in incidence can produce no additional lift. In most aeroplane wings, an increase in incidence beyond the optimum angle causes the wing to lift less at the same air speed. Now if the aeroplane in such an attitude roll, the increased angle of incidence of the downward moving wing gives no more lift on that wing and hence the rolling is unresisted. The damping of the roll will be zero, or even negative. In the Curtiss aeroplane, the low speed chosen required an incidence of 15°.5, very near the "burble point," or angle of maximum lift for the wings. The small value -78 of L_p appears to be one of the principal causes of the instability. In the Clark model, the wing loading is smaller and an equal speed about 44 miles per hour is obtained for an incidence of only 6°, giving $L_p = -319$. The lowest speed of the Clark model is taken as about 37 miles per hour where an incidence of but 12° is needed. L_p at this angle is -224.

It appears that lateral dynamical stability is incompatible with a high wing loading which requires a large angle at landing speed. The analysis of longitudinal stability led to a similar conclusion.

If we turn to practical aviation we observe that aeroplanes which are noted for their steadiness at low speeds are the light Antoinette, Farman, and the various German Taubes derived from the Etrich. All these aeroplanes have large wing area and light loading, probably between 3 and 4 pounds lift per square foot. The light loading enables these aeroplanes to gain a safe low speed without having the angle of incidence near the angle of maximum lift.

In the Clark model the loading is about 3.55 pounds per square foot, while it is 5.2 in the Curtiss type discussed. More recently the

Curtiss has been given greater wing area in order to reduce the loading. It should be stated that the comparison is not quite fair, since the total weight of the Clark aeroplane was taken as 1,600 pounds which includes only half the full 5.6 hours' gasolene supply. However, the advantage of light wing loading is more clearly brought out by the marked difference in weight per square foot wing area.

The following table summarizes all the information available and may be used to make further comparisons if desired:

	Clark Tractor[2]			Curtiss Tractor[2]		Blériot Monoplane[3]
Wing area	464.0	.....	.....	384.0	.....	244.0
Mean span	40.2	.....	.....	36.0	.....	.....
Mean chord	5.77	.....	.....	5.3	.....	.....
Mean gap	6.37	.....	.....	5.3	.....	.....
Area, fixed tail	16.1	.....	.....	23.0	.....	.....
Area, elevators	16.0	.....	.....	19.0	.....	.....
Area, rudder	9.35	.....	.....	7.8	.....	.....
Length, body	24.5	.....	.....	26.0	.....	.....
Weight, lbs	1600	.....	.....	1800	.....	1800
Rise of wing	1°.63	.....	.....	0°	.....	1°.8
Lbs. per sq. ft	3.55	.....	.....	5.2	.....	7.38
Angle of incidence	0°	6°.0	12°.0	1°.0	15°.5	6°.0
V, miles, hour	76.9	44.6	36.9	78.9	43.6	65.0
U, ft.-seconds	−112.5	− 65.3	− 54.0	−115.5	− 63.8	− 95.4
m	50.0	.....	.....	56.0	.....	56.0
K_A, feet	5.2	.....	.....	6.06	.....	5.0
K_C, feet	6.975	.....	.....	8.4	.....	6.0
Y_v	− .204	− .0878	− .106	− .248	− .09	− .108
L_v	+ 3.06	+ 3.44	+ 1.91	+ .844	+ 2.7	+ .70
N_v	− .449	− .351	− .53	− .894	− .45	− .44
Y_p	0	0	0	0	0	0
L_p	−631.0	−319.0	−224.0	−314.0	− 78.0	−167.0
N_p	0	+ 33.5	+ 57.0	0	+ 37.7	+ 24.0
Y_r	0	0	0	0	0	0
L_r	+ 77.0	+132.5	+160.0	+ 55.2	+101.0	+ 54.0
N_r	− 39.4	− 26.0	− 38.9	− 27.0	− 30.4	− 31.0
A_2	1310.0	1310.0	1310.0	2590.0	2590.0	900.0
B_2	31800.0	16350.0	12090.0	23800.0	6860.0	6780.0
C_2	32700.0	5910.0	1630.0	18000.0	209.0	5580.0
D_2	41780.0	5490.0	3490.0	34600.0	5590.0	6640.0
E_2	2770.0	1386.0	−335.0	−855.0	1175.0	− 68.0
Routh's discr	37×10^{12}	12×10^{10}	4×10^{9}	9×10^{12}	-7×10^{10}	21.5×10^{10}
Spiral Motion						
Damp 50% in, sec.	10.4	2.7	.....	.....	3.3	.....
Double in, sec.	.....	.....	7.2[1]	28.0[1]	.....	68.0[1]
Rolling						
Damp 50% in, sec.	.03	.06	.076	.08	.26	.10
"Dutch Roll"						
Period, sec	5.9	10.7	12.0	5.24	5.7	6.5
Damp 50% in, sec.	1.4	1.3	5.95	1.77	.....	1.65
Double in, sec	.....	.....	.....	.....	7.66[1]	.....

[1] Unstable.
[2] Tested at Mass. Institute of Technology, Boston.
[3] Tested at Teddington, England.